Robust Portfolio Choice and Indifference Valuation*

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Abstract

We solve, theoretically and numerically, the problems of optimal portfolio choice and indifference valuation in a general continuous-time setting. The setting features (i) ambiguity and time-consistent ambiguity averse preferences, (ii) discontinuities in the asset price processes, with a general and possibly infinite activity jump part next to a continuous diffusion part, and (iii) general and possibly non-convex trading constraints. We characterize our solutions as solutions to Backward Stochastic Differential Equations (BSDEs). Generalizing Kobylanski's result for quadratic BSDEs to an infinite activity jump setting, we prove existence and uniqueness of the solution to a general class of BSDEs, encompassing the solutions to our portfolio choice and valuation problems as special cases. We provide an explicit decomposition of the excess return on an asset into a risk premium and an ambiguity premium, and a further decomposition into a piece stemming from the diffusion part and a piece stemming from the jump part. We further compute our solutions in a few examples by numerically solving the corresponding BSDEs using regression techniques.

Keywords: Robust preferences; Ambiguity aversion; Recursiveness and time-consistency; Convex risk measures; Portfolio choice; Incomplete markets; Jumps; Indifference valuation; BSDEs; Exponential and power utility.

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1 Introduction

Two main problems in asset pricing are portfolio choice and valuation in incomplete markets. The study of the dynamic portfolio choice problem goes back to Merton [64, 65] who approached it using stochastic control theory. It has since been considered by numerous authors in a wide variety of settings. Contributions relevant to the setting considered in this paper include Cvitanic and Karatzas [21], who prove existence and uniqueness of the solution (optimal portfolio) to the utility maximization problem in a Brownian filtration when restricting investment strategies to convex sets; and Kallsen [52], who solves the continuous-time utility maximization problem in a market where asset prices follow exponential Lévy processes; both using the duality or martingale approach. For further references, see the review of Schachermayer [80].

A widely adopted method for valuation in incomplete markets is indifference valuation (Carmona [14]). It is related to the portfolio choice problem: under indifference valuation, the price of a claim is such that the agent is indifferent between selling and not selling the claim, provided that each of the two alternatives is combined with an optimal portfolio choice that maximizes utility. Particularly popular is exponential indifference valuation due to its analytical tractability on the one hand — the exponential form induces a convenient translation invariance property — and its theoretically appealing properties, especially in a dynamic context, on the other (El Karoui and Rouge [30], Delbaen et al. [22], Kabanov and Stricker [51], Mania and Schweizer [63]). See also Hu, Imkeller and Müller [47], Becherer [5], Morlais [67, 68], and Cheridito and Hu [17] for recent work on portfolio choice.

Many decision-making problems, including asset pricing problems, involve ambiguity (probabilities unknown) and it is important to distinguish them from decision-making problems under risk (probabilities given). A rich class of models for decision-making under ambiguity is that of variational preferences (Maccheroni, Marinacci and Rustichini [61]). It includes the popular maxmin expected utility of Gilboa and Schmeidler [37], also referred to as multiple priors, and the multiplier preferences of Hansen and Sargent [42, 43] as special cases. Under ambiguity, the true probabilistic model is unknown to the decision maker (model uncertainty); approaches that explicitly account for the possibility that a specific probabilistic model may not be correct but only an approximation, are commonly referred to as robust approaches. Such robust approaches are both normatively (prescriptively) appealing and empirically (descriptively) relevant. But for our results that follow it is unimportant which point of view one takes, whether prescriptive or descriptive.

Recently, there has been a growing interest in the effects of ambiguity on portfolio choice and valuation; see, for example, Chen and Epstein [15], Lazrak and Quenez [57], Maenhout [62], Müller [69], Schied [81], Klöppel and Schweizer [54], Föllmer, Schied and Weber [34], Owari [71] and Sircar and Sturm [82]. The importance of incorporating ambiguity in the problems of portfolio choice and valuation is not merely theoretical as ambiguity plays a potential role in addressing important failures of purely risk-based settings that rule out model uncertainty. Examples of such failures include the equity premium puzzle and the home-bias puzzle (Chen and Epstein [15]). However, all above-mentioned papers featuring ambiguity are restricted to a continuous Brownian setting and do not allow for any discontinuities (jumps) in the asset price processes.

In this paper we solve, theoretically and numerically, the two canonical optimization problems of portfolio choice and indifference valuation, under ambiguity and time-consistent ambiguity averse preferences, and in a further general continuous-time setting: besides a continuous diffusion component, we allow for a general and possibly infinite activity jump component in the asset price processes, and for general and possibly non-convex trading constraints regarding buying and short-selling. Note that by the nature of jumps, the jump component of a semi-martingale asset price dynamics model is not unlikely to be exposed to model uncertainty and with such a model a setting allowing for ambiguity and ambiguity averse preferences seems particularly appealing.

As regards the ambiguity averse preferences, we assume in particular that the economic agent exhibits recursive variational preferences, with linear or logarithmic utility, or recursive multiple priors, with power or exponential utility. The reason that in the case of power or exponential utility the analysis necessarily needs to be restricted to multiple priors, is that otherwise the preferences do not exhibit recursiveness, also known as Bellman's principle, which is crucial to characterize and compute the optimal solutions. We note that, with linear utility, recursiveness is equivalent to time-consistency. For the use of this or similar notions of time-consistency, see, among many others, Duffie and Epstein [28], Chen and Epstein [15], Epstein and Schneider [31] and Ruszczyński and Shapiro [78]. To illustrate the generality of the ambiguity averse preferences considered in this paper, we explicate that specific examples include the relative entropy (Kullback-Leibler divergence), a discrete set of dynamic worst case scenarios, mean and variance of the underlying asset known, mean known to lie in a certain interval, and ball-robustification.

We prove that the solutions to the optimal portfolio choice and (in the case of linear or exponential utility) valuation problems can be characterized as solutions to backward stochastic differential equations (BSDEs). As a by-product, which is of interest in its own right, we prove existence and uniqueness of the solution to the general class of BSDEs with jumps and having a drift (or driver) that grows at most quadratically, encompassing the solutions to our portfolio choice and valuation problems as special cases. Essentially, this by-product generalizes the important existence and comparison results obtained by Kobylanski [55], for BSDEs with at most quadratic growth in a Brownian filtration, to a general and possibly infinite activity jump setting. We also provide an economic interpretation to the optimal solutions and to the excess return on an asset, which we explicitly decompose into a risk premium and an ambiguity premium, and further decompose into a piece stemming from the diffusion part and a piece stemming from the jump part. We finally provide a numerically tractable procedure to compute our solutions (by numerically solving the corresponding BSDEs using regression techniques) and implement this procedure in a few examples.

A BSDE may be seen as a dynamic programming principle in a continuous-time stochastic setting. BSDEs play an important role in stochastic control; see, for example, Pardoux and Peng [72], Duffie and Epstein [28], El Karoui, Peng and Quenez [29], Chen and Epstein [15], Lazrak and Quenez [57], Skiadas [83], Lim [59, 60], Hamadène and Jeanblanc [41], Horst and Müller [44], and also the early work of Bismut [11]. In a Markovian setting, BSDEs correspond to semi-linear PDEs. As is well-known, the solution to a utility maximization problem with a numerical preference representation specified by a BSDE can in turn be characterized as a solution to a BSDE; see Klöppel and Schweizer [54] and Sircar and Sturm [82] for recent applications of this technique to portfolio choice and indifference valuation in a purely Brownian setting. Therefore, a standard approach in utility maximization has been to try converting the utility maximization problem into a 'BSDE type' stochastic control problem. One of the advantages of this approach to portfolio choice is that, contrary to static duality methods, BSDEs can also deal with non-convex trading constraints. Another advantage of using BSDEs is that their solutions can be efficiently computed numerically by Monte Carlo simulation.

Applications of BSDEs to utility maximization problems in incomplete markets in a Brownian setting include (with exponential, logarithmic or power utility) Hu, Imkeller and Müller [47], Cheridito and Hu [17], and (with a general utility function) Horst et al. [45]; for a setting with continuous filtration or non-continuous filtration (but with exponential utility), see Mania and Schweizer [63], Morlais [67] and Becherer [5]. Morlais [68] generalizes some of these results adopting an exponential utility function and allowing for infinite activity jumps in the asset price processes, in a purely risk-based setting without ambiguity. In particular, she proves existence and uniqueness results for a special quadratic BSDE. Mathematically, we generalize parts of her and Becherer's [5] results by proving existence and uniqueness results for all possibly infinite activity jump BSDEs with a driver function that grows at most quadratically. Contrary to Morlais [68], Becherer [5] and Kobylanski [55], who prove their results by solving the primal problem, we use a duality approach, generalizing parts of the methods developed by Delbaen, Hu and Bao [24] in a Brownian filtration to a setting with possibly infinite activity jumps.

There are only few works studying the portfolio choice and valuation problems in a setting with jumps *and* ambiguity. Bordigoni, Matoussi and Schweizer [13] study ambiguity using the relative entropy and Jeanblanc, Matoussi and Ngoupeyou [50] generalize this work to a non-continuous filtration, assuming a one-point jump distribution. Björk and Slinko [12] study asset prices with jumps in a non-utility framework, using different possible pricing kernels to obtain good-deal bound prices. Independently of our work, Delong [25] and Øksendal and Sulem [70] have recently also considered model uncertainty in continuous-time jump settings. In Delong [25], the portfolio choice problem is solved in a setting with a degenerate, one-point jump distribution in case of a linear utility and multiple priors preferences. Øksendal and Sulem [70] use a generalization of multiple priors different from variational preferences. Assuming that the solutions of certain BSDEs exist and that comparison principles hold, they derive optimality conditions using techniques different from ours. Besides they (can) only characterize the optimal solution in the case of deterministic jump coefficients. Not only do we characterize the solutions to the portfolio choice and indifference valuation problems in a general continuous-time setting, we also provide new existence and uniqueness results for solutions to the corresponding class of BSDEs. We are not aware of other work on the problems of portfolio choice and indifference valuation that allows for a comparable degree of generality for all these features — ambiguity, jumps, and general trading constraints — together.

To sum up:

- There are not many works to date that consider both ambiguity and jumps in portfolio choice and valuation. If ambiguity and jumps are considered, then typically the analysis is restricted to degenerate, one-point jump distributions and the preferences that are treated are less general than those analyzed in our paper.

- To our best knowledge, we are the first to provide a complete solution to the portfolio choice and indifference valuation problems in a general and possibly infinite activity jump setting under time-consistent ambiguity averse preferences, proving existence and comparison results for the corresponding BSDEs.

- Since we do not rely on saddle point techniques (except in the case of power utility) we are able to solve the portfolio choice problem also for possibly non-convex (but compact) trading constraints.

- Mathematically, our contribution is to show that BSDEs with a driver that grows at most quadratically have unique solutions. This generalizes Kobylanski's [55] results on the existence of solutions to quadratic BSDEs (arguably one of the main results in the BSDE literature) to an infinite activity jump setting.

It is known that in a setting without ambiguity, discontinuities (jumps) in the asset price process have a discernible impact on the optimal portfolio choice (Kallsen [52], Aït-Sahalia, Cacho-Diaz and Hurd [1]). This impact is especially prevalent when allowing for dependencies between the jumps, limiting the benefits of international diversification, whence providing a (further) possible explanation for the empirically observed home-bias in investors' portfolios. The impact of constraints on buying and short-selling is documented in a rich literature (see Rubinstein [76] for a review); it is important to account for such trading constraints in the most general fashion.

This paper is organized as follows. In Section 2, we introduce the basic setting and review some preliminaries for BSDEs. In Section 3, we specify and characterize in detail the economic agent's preferences. In Section 4, we state the dynamic optimization problems, characterize their solutions and prove existence and uniqueness of these solutions. Section 5 examines the decomposition of the excess return on an asset. Section 6 discusses and illustrates the numerical implementation in some examples. Proofs are collected in the Appendix.

2 Setting and Preliminaries

2.1 Asset Return Dynamics, Trading Constraints and Preferences

We consider an economic agent with initial wealth w_0, which he can invest in a risk-less bond and risky assets. At a given maturity time $T > 0$, the agent is endowed with an additional payoff F. A classical problem in asset pricing is the question of how the agent should determine his optimal investment strategy. To answer this question, one first needs to address the following issues: (i) How to model the dynamics of the risky assets? (ii) Which constraints to impose on the trading strategies allowed? (iii) How to evaluate the quality of the agent's investment strategy? This section describes our approach to these issues.

For the dynamics of the risky assets, we assume a continuous-time setting with a general and possibly infinite activity jump component next to a general continuous diffusion component with stochastic volatility, and ambiguity. Large jumps in asset prices represent major financial economic shocks, such as market crashes, shocks resulting from unexpected announcements of the FED, or environmental disasters causing sudden movements in prices. Ambiguity, which is also referred to as model uncertainty, means that the 'true' probabilistic model is unknown to the decision maker. A setting featuring ambiguity seems particularly appealing when allowing for jumps in asset prices: large jumps are inherently rare and the jump component of the model may therefore easily be subject to model uncertainty.

Formally, we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$. Throughout, the dependence of random variables, stochastic processes, predictable functionals, counting measures, sets and subdifferentials on ω will be suppressed whenever possible. We assume that the probability space is equipped with two independent stochastic processes:

(i) A standard d-dimensional Brownian motion W.

(ii) A real-valued marked point process p on $[0,T]\times\mathbb{R}\setminus\{0\}$. We denote by $N_p(ds,dx)$ the associated random (counting) measure. We assume its compensator (or mean or intensity measure) $\hat{N}_p(ds,dx)$ to be of the form

$$\hat{N}_p(ds,dx)=n_p(dx)ds.$$

We suppose that the measure $n_p(dx)$ is non-negative and satisfies for every $\epsilon>0$, $n_p(\mathbb{R}\setminus\{(-\epsilon,\epsilon)\})<\infty$. Furthermore,

$$\int_{\mathbb{R}\setminus\{0\}}(|x|^2\wedge 1)n_p(dx)<\infty.$$

Throughout, equalities and inequalities between random variables are meant to hold P-almost surely (a.s.); two random variables are identified if they are equal P-a.s. We denote by $L^0(n_p)$ the space of $\mathcal{B}(\mathbb{R}\setminus\{0\})$-measurable functions. Equalities and inequalities between functionals of $L^0(n_p)$ are meant to hold $n_p(dx)$-a.s., and two elements of $L^0(n_p)$ are identified if they are equal $n_p(dx)$-a.s. Furthermore, for any t, inequalities between $\mathcal{F}_t\otimes L^0(n_p)$-measurable random functionals are meant to hold $dP\times n_p(dx)$-a.s.; two elements are identified if they are equal $dP\times n_p(dx)$-a.s. Similarly, inequalities between stochastic processes are meant to hold $dP\times ds$-a.s. For functionals $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ and $h:\mathbb{R}\setminus\{0\}\to\mathbb{R}$, define $f\cdot h:=\int_{\mathbb{R}\setminus\{0\}}f(x)h(x)n_p(dx)$. Furthermore, for $H:\mathbb{R}\setminus\{0\}\to\mathbb{R}^k$ with $H=(h^1,\dots,h^k)$, let $f\cdot H:=(f\cdot h^1,\dots,f\cdot h^k)$.

We assume that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is the completion of the filtration generated by W and N_p. We denote by $\mathcal{P}$ the predictable σ-algebra on $[0,T]\times\Omega$ with respect to $(\mathcal{F}_t)$. Let $\tilde{N}_p(ds,dx):=N_p(ds,dx)-\hat{N}_p(ds,dx)$. Financial economic shocks arrive at discrete points in time. Every shock comes with a 'marker' x. $N_p(s,dx)$ is one if there is a shock at time s with marker x and $n_p(dx)ds$ is the expected number of shocks with size 'around' x per time unit 'around' time s.

Remark 2.1 All our results also hold for a general counting measure with a predictable compensator taking the form $n_p(s,\omega,dx)ds$ with multi-dimensional markers.

We assume that the financial market consists of a risk-free bond with interest rate normalized to zero, and $n\leq d$ stocks. The price process of stock i, denoted by S^i, evolves according to the semi-martingale dynamics

$$\frac{dS^i_t}{S^i_{t-}}=b^i_tdt+\sigma^i_tdW_t+\int_{\mathbb{R}\setminus\{0\}}\beta^i_t(x)\tilde{N}_p(dt,dx),\qquad i=1,\dots,n,\tag{2.1}$$

where b^i (σ^i, β^i) are $\mathbb{R}$ ($\mathbb{R}^{1\times d}$, $\mathbb{R}$)-valued, predictable and uniformly bounded stochastic processes. b^i is commonly referred to as the excess return: the holder of a risky asset should be compensated for the risk he is bearing. The second term on the right-hand side (RHS) of (2.1) represents noise due to 'normal' market movements and is locally Gaussian. We assume that σ has full rank and $\sigma\sigma^\intercal$ is uniformly elliptic, i.e., $\varepsilon I_n\preceq\sigma_t\sigma_t^\intercal\preceq\hat{K}I_n$, for some constants $\hat{K}>\varepsilon>0$. The third term on the RHS of (2.1) represents the dynamics due to financial economic shocks. $\beta^i(x)$ is the impact (jump size) of a shock with 'marker' x on the asset price S^i. We assume that β^i is larger than -1 to ensure positivity of S^i, $i = 1, \ldots, n$. We further assume that $\beta^i \in L^{2,\infty}$, $i = 1, \ldots, n$, where

$$L^{2,\infty} = \left\{ \tilde{H} | \tilde{H} \text{ is } \mathcal{P} \otimes \mathcal{B}(\mathbb{R} \setminus \{0\}) \text{ measurable and } \left\| \sup_s \int_{\mathbb{R}\setminus\{0\}} |\tilde{H}_s(x)|^2 n_p(dx) \right\|_\infty < \infty \right\};$$

$||\cdot||_\infty$ denotes the norm given by the (essential) supremum over all ω. This condition is satisfied, for instance, if either n_p is finite, or if for all $|x|$ small and for all s, $|\beta_s(x)| \leq K|x|$. If $n < d$, the market is incomplete. If $n = d$, the market is typically still incomplete because of the jump component of the model. Our model includes all Lévy processes with finite or infinite jump activity. In recent years, the latter case has often been adopted in financial engineering. As b, σ, and β in (2.1) can all depend on ω and s, our results also hold for processes that do not have independent increments. In particular, our setting also includes the case of stochastic volatility and / or stochastic jump rates as long as our boundedness and integrability conditions are met.

In the special case that $n_p(\mathbb{R} \setminus \{0\}) < \infty$ holds, so that we only have finitely many jumps, S may be written as a standard jump diffusion model, originating from Merton [66]. In this restricted model, asset returns evolve according to

$$\frac{dS_t^i}{S_{t-}^i} = b_t^i dt + \sigma_t^i dW_t + \sum_l I_{\{T_l^i = t\}} J_l^i, \qquad i = 1, \ldots, n,$$

for jump times $T_1^i, T_2^i, \ldots$ with corresponding jump sizes $J_1^i, J_2^i, \ldots$.

For $i = 1, \ldots, n$, the process π_t^i represents the amount of capital invested in stock i at time t, and the number of shares is $\frac{\pi_t^i}{S_t^i}$. The wealth process $X^{(\pi)}$ of a predictable trading strategy π with initial capital w_0 satisfies

$$X_t^{(\pi)} = w_0 + \sum_{i=1}^n \int_0^t \frac{\pi_u^i}{S_{u-}^i} dS_u^i = w_0 + \int_0^t \pi_u(\sigma_u dW_u + b_u du) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \pi_u \beta_u(x) \tilde{N}_p(du, dx).$$

We assume that the agent is allowed to choose trading strategies taking values in a compact and possibly non-convex set $\Pi \subset \mathbb{R}^{1\times n}$ a.s. We call π an admissible trading strategy if it is predictable and takes values only in Π. We denote the set of all admissible trading strategies by $\mathcal{A}$. Since the set Π is compact, for every trading strategy π, the wealth process $\sup_t |X_t^{(\pi)}|$ is square-integrable. By our assumptions, there exists a local martingale measure Q^f under which $W_t - \int_0^t \sigma_s^\intercal (\sigma_s \sigma_s^\intercal)^{-1} b_s ds$ is a Brownian motion (and hence S is a local martingale). In particular, there is no arbitrage in the market.

The agent, choosing an investment strategy (π_t) and being endowed with a payoff F, eventually holds the portfolio $F + X_T^{(\pi)}$ at maturity. The final issue to be addressed is which decision criterion to use when evaluating the quality of the agent's portfolio choice. The classical decision criterion in a setting featuring ambiguity is Savage's [79] subjective expected utility; it postulates that the economic agent specifies a subjective probability measure P and a utility function u, and evaluates the portfolio according to $U(F + X_T^{(\pi)}) = \mathrm{E}\left[u(F + X_T^{(\pi)})\right]$. We note that specifying the measure P in our setting implies specifying (estimating) the excess return b_t, the Gaussian volatility σ_t, and the impact of the jumps $\beta_t(x) n_p(dx)$: a challenging econometric exercise. From a normative point of view, it is appealing to consider instead a robust decision criterion, which makes sure that the portfolio choice accounts for a class of potential probabilistic models and is not based on just one single model. Furthermore, also empirically it is well-known in decision theory that, faced with ambiguity, agents make decisions that are incompatible with subjective expected utility.

Various alternative approaches to decision-making under ambiguity have emerged in the literature. Among the best-known alternatives is multiple priors, of Gilboa and Schmeidler [37] (see also Wald [85] and Huber [48]); it postulates that an economic agent evaluates his portfolio according to $U(F+X_T^{(\pi)}) = \inf_{Q\in M} \mathrm{E}_Q[u(F+X_T^{(\pi)})]$, where M is a set of probabilistic models (or priors). Multiple priors was significantly generalized by Maccheroni, Marinacci, and Rustichini [61] to the theory of variational preferences, postulating that an economic agent evaluates his portfolio according to

$$U(F+X_T^{(\pi)}) = \inf_{Q\in\mathcal{Q}} \{\mathrm{E}_Q[u(F+X_T^{(\pi)})] + c(Q)\}. \tag{2.2}$$

Variational preferences go beyond multiple priors preferences by allowing to attach a plausibility index c (the penalty function) to every probabilistic model Q in the class of probabilistic models $\mathcal{Q}$ under consideration. If $c(Q) = \infty$, the minimum in (2.2) is not attained in this particular Q, meaning that probabilistic models with infinite penalty are considered fully unreliable and are effectively excluded from the analysis. Multiple priors occurs when $c(Q) = I_M$, the penalty function that is zero if $Q \in M$ and ∞ otherwise, attaching the same plausibility to all probabilistic models in M. In the case that u is linear, multiple priors corresponds to coherent risk measures (Artzner et al. [2]) and variational preferences corresponds to convex risk measures (Ben-Tal and Teboulle [8, 9], Föllmer and Schied [32], Frittelli and Rosazza Gianin [35], Ruszczyński and Shapiro [77, 78]); see Laeven and Stadje [56] for further results on these connections.

To distinguish between $U(\cdot)$ and $u(\cdot)$, we call U an evaluation and u a utility function. To approach the portfolio choice and valuation problems using a dynamic programming principle, we (need to) consider the dynamic version of (2.2), which is given by

$$U_t(F+X_T^{(\pi)}) := \operatorname{ess\,inf}_{Q\in\mathcal{Q}} \{\mathrm{E}_Q[u(F+X_T^{(\pi)})|\mathcal{F}_t] + c_t(Q)\}, \tag{2.3}$$

in which $c_t(Q)$ reflects the esteemed plausibility of the model Q given the information up to time t. The portfolio choice problem is then finally given by

$$V_t(F) = \max_{\pi\in\mathcal{A}} U_t(F+X_T^{(\pi)}),$$

at time t.

The class of all alternative probabilistic models considered is specified as

$$\mathcal{Q} = \{Q | \text{ If for an event } A: \; P(A) = 0, \text{ then also } Q(A) = 0\} = \{Q | Q \ll P\},$$

i.e., all measures Q that are absolutely continuous with respect to the reference model P are considered; sets with probability zero under the reference model P still have probability zero under the alternative model Q. It means, for example, that if, with probability one under P, the financial asset has only finitely many jumps, it also has only finitely many jumps under every Q. We subsequently assume that u is linear, exponential, power or logarithmic. In the case that u is linear or exponential, the problem is translation invariant, in particular, the optimal solution will be independent of the wealth of the agent. In this case, we can also explicitly calculate the indifference valuation. We further consider essentially all plausibility indices that induce recursive preferences; see Section 3 for further details, and illustrations with specific examples.

Remark 2.2 Notice that the conditional expectation in (2.3) is, in fact, defined only Q-a.s., while equations and the essential infimum need to be defined P-a.s. Therefore, to avoid cumbersome notation and to ensure that conditional expectations are defined P-a.s., we will throughout, for a stopping time σ and a random variable F, denote $\mathrm{E}_Q[F|\mathcal{F}_\sigma] = \mathrm{E}[\xi_{\sigma,T}F|\mathcal{F}_\sigma]$, with $\xi_{\sigma,T} = \frac{D_T^Q}{D_\sigma^Q}$ where the Radon-Nikodym density $D_\sigma^Q := \mathrm{E}\left[\frac{dQ}{dP}|\mathcal{F}_\sigma\right] > 0$ is strictly positive, and $\xi_{\sigma,T} = 1$ whenever $D_\sigma^Q = 0$; see also Cheridito and Kupper [18], who have used a similar notation when dealing with dynamic monetary utility functions.

2.2 BSDEs

We solve the portfolio choice and indifference valuation problems using backward stochastic differential equations (BSDEs). We denote by $|\cdot|$ the Euclidean norm and by $\mathcal{S}^\infty$ the class of all one-dimensional $(\mathcal{F}_t)$-adapted semi-martingales X which are bounded. Define $|X|_{S^\infty} = \|\sup_t |X_t|\|_\infty$. Let

$$\mathcal{H}^2 := \left\{ Z = (Z^1, \ldots, Z^d) \middle| Z^i \in \mathcal{P} \text{ for } i = 1, \ldots, d \text{ and } \mathrm{E}\left[\int_0^T |Z_s|^2 ds\right] < \infty \right\}.$$

Furthermore, we denote by $L^2(dP \times n_p(s, dx) \times ds)$ all functions measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$ which are square-integrable with respect to $dP \times n_p(dx) \times ds$.

Consider a function

$$\begin{array}{lccccccccc} g: & [0,T] & \times & \Omega & \times & \mathbb{R}^d & \times & L^2(n_p) & \to & \mathbb{R} \\ & (t, & & \omega, & & z, & & \tilde{z}) & \longmapsto & g(t, \omega, z, \tilde{z}). \end{array}$$

To simplify the notations, we will often write $g(t, z, \tilde{z})$ instead of $g(t, \omega, z, \tilde{z})$. Furthermore, we will usually write $g(t, \cdot, \cdot)$ instead of $g(t, \omega, \cdot, \cdot)$. Notice that this is consistent with suppressing the ω argument when considering random variables or stochastic processes.

A solution to the (one-dimensional) BSDE with driver g mapping to $\mathbb{R}$ and terminal condition $F \in L^\infty(\mathcal{F}_T)$ is a triple of processes $(Y, Z, \tilde{Z}) \in \mathcal{S}^\infty \times \mathcal{H}^2 \times L^2(dP \times n_p(dx) \times ds)$ such that

$$dY_t = g(t, Z_t, \tilde{Z}_t)dt - Z_t dW_t - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_t(x)\tilde{N}_p(dt, dx) \text{ and } Y_T = F.$$

Often times BSDEs are written in the following equivalent form:

$$Y_t = F - \int_t^T g(s, Z_s, \tilde{Z}_s)ds + \int_t^T Z_s dW_s + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x)\tilde{N}_p(ds, dx).$$

Since the terminal condition is given at maturity time T, BSDEs have to be computed backwards in time, whence their name. As in many applications a terminal reward is specified, and solutions of BSDEs satisfy a dynamic programming principle, BSDEs are often applied to solve problems in stochastic control and mathematical finance; see the references provided in the Introduction.

It is well-known that if $g(t,0,0)$ is in $L^\infty(dP\times dt)$ and g is additionally uniformly Lipschitz continuous, that is, there exists $K>0$ such that for all t,

$$|g(t,z_1,\tilde{z}_1)-g(t,z_0,\tilde{z}_0)|\leq K\left(|z_1-z_0|+\sqrt{\int_{\mathbb{R}\setminus\{0\}}|\tilde{z}_1(x)-\tilde{z}_0(x)|^2 n_p(dx)}\right),$$

then a unique solution to the corresponding BSDE exists; see, for example, Royer [75]. However, in the case of exponential or power utility, the BSDEs we will encounter below have drivers that grow quadratically, and, in the case of linear or logarithmic utility, for penalty functions 'growing slowly', the corresponding BSDEs will also not be Lipschitz continuous. Therefore, new analytical and numerical tools need to be developed.

Example 2.3 Let F be a bounded payoff and define $Y_t=\mathrm{E}\left[F|\mathcal{F}_t\right]$. Then, $Y_T=F$. Moreover, by the martingale representation theorem (see, e.g., Jacod and Shiryaev [49], Ch. 3, Sec. 4) there exist predictable processes Z and $\tilde{Z}$ such that Y satisfies

$$dY_t=-Z_t dW_t-\int_{\mathbb{R}\setminus\{0\}}\tilde{Z}_t(x)\tilde{N}_p(dt,dx).$$

This is the simplest BSDE with $g=0$.

Hence, a conditional expectation may be seen as a BSDE with $g=0$. It explains why BSDEs are also being referred to as g-expectations. The name should express that a BSDE may be viewed as a generalized (usually non-linear) conditional expectation with an additional drift.

Example 2.4 Let F be a bounded payoff and define $Y_t=\mathrm{E}_{Q^f}\left[F|\mathcal{F}_t\right]$; recall that Q^f is the local martingale measure under which $W_t-\int_0^t\sigma_s^\intercal(\sigma_s\sigma_s^\intercal)^{-1}b_s ds$ is a Brownian motion. Then, by the martingale representation theorem and the Lenglart-Girsanov theorem (Jacod and Shiryaev [49], Ch. 3, Th. 3.11), Y satisfies

$$dY_t=-Z_t\sigma_t^\intercal(\sigma_t\sigma_t^\intercal)^{-1}b_t dt-Z_t dW_t-\int_{\mathbb{R}\setminus\{0\}}\tilde{Z}_t(x)\tilde{N}_p(dt,dx)\text{ and }Y_T=F.$$

This is a linear BSDE with $g(t,z,\tilde{z})=-z\sigma_t^\intercal(\sigma_t\sigma_t^\intercal)^{-1}b_t$.

In a Markovian setting, g-expectations correspond to viscosity solutions to semi-linear parabolic PDEs (or PIDEs in the case of jumps); see, for example, El Karoui, Peng and Quenez [29] in a Brownian setting and Barles, Buckdahn and Pardoux [4] in the case of jumps. However, because BSDEs are more general (since they do not rely on a Markovian structure) and because our numerical implementation is based on Monte Carlo simulations which are again based on the structure of the BSDE, we will approach the problems under consideration using BSDEs rather than using PIDEs.

3 Ambiguity Averse Preferences and Dynamic Programming

We specify below assumptions on the plausibility index c in (2.2)-(2.3), which essentially correspond to assuming that the agent exhibits recursive preferences. But let us first take a closer look at the set of alternative models $\mathcal{Q} = \{Q | Q \ll P\}$. It is well-known that in a Brownian filtration, every probability measure Q absolutely continuous with respect to P can be identified with a stochastic drift $q : [0,T] \times \Omega \to \mathbb{R}^d$ such that $W_t - \int_0^t q_s ds$ is a Brownian motion under Q. It means that in a Brownian filtration, the setting of ambiguity, in which a collection of probability measures (priors) is considered rather than a single probability measure, can be fully described by a collection of drifts q. Note that the dependency of W and q on ω is suppressed.

Now let us address the question of how to model ambiguity with respect to the jump component of the model. If $Q \in \mathcal{Q}$, we denote by D_t the Radon-Nikodym derivative $D_t = \mathrm{E}\left[\frac{dQ}{dP} | \mathcal{F}_t\right]$ and define $\tau = \inf\{t | D_t = 0\} \wedge T$. One can show that there exist a predictable stochastic drift q and a random function $\psi : [0,T] \times \Omega \times \mathbb{R} \setminus \{0\} \to [-1, \infty)$, measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$, such that the Radon-Nikodym derivative can be written as

$$D_t = \exp\left\{ \int_0^t q_s dW_s - \frac{1}{2}\int_0^t |q_s|^2 ds + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \psi_s(x) \tilde{N}_p(ds, dx) \right. \\ \left. + \int_0^t \int_{\mathbb{R}\setminus\{0\}} [\log(1+\psi_s(x)) - \psi_s(x)] N_p(ds, dx) \right\}, \tag{3.1}$$

for $t \leq \tau$. In particular, Q is uniquely characterized by q and ψ.

Expression (3.1) is seen as follows: Clearly, the Radon-Nikodym derivative D_t is a martingale. For $t \geq \tau$, we must have that $D_t = 0$. Furthermore, the whole path of D_{t-} is strictly positive up to time τ, see Lemma A.19 in the Appendix. By Jacod and Shiryaev [49], Ch. 3, Sec. 4, there exist a locally integrable (see Definition A.2 in the Appendix) process $H : [0,T] \times \Omega \to \mathbb{R}$, measurable with respect to $\mathcal{P}$, and a locally integrable function $\tilde{H} : [0,T] \times \Omega \times \mathbb{R} \setminus \{0\} \to \mathbb{R}$, measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R} \setminus \{0\})$, such that

$$dD_t = H_t dW_t + \int_{\mathbb{R}\setminus\{0\}} \tilde{H}_t(x) \tilde{N}_p(t, dx),$$

with $D_0 = 1$. For $t \geq \tau$, we must have that $H_t = 0$ and $\tilde{H}_t = 0$. Therefore, defining $0/0 = 0$, we obtain

$$\begin{aligned} dD_t &= D_{t-} \left(\frac{H_t}{D_{t-}} dW_t + \int_{\mathbb{R}\setminus\{0\}} \frac{\tilde{H}_t(x)}{D_{t-}} \tilde{N}_p(dt, dx) \right) \\ &=: D_{t-} \left(q_t dW_t + \int_{\mathbb{R}\setminus\{0\}} \psi_t(x) \tilde{N}_p(dt, dx) \right), \end{aligned} \tag{3.2}$$

for $t \leq \tau$. The solution to this SDE is given by the stochastic exponential $\mathcal{E}\left(\int_0^t q_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \psi_s(x) \tilde{N}_p(ds, dx)\right)$, which equals the right-hand side in (3.1). The stochastic exponential is also referred to as the Doléans-Dade exponential.

Since D_t is non-negative, we must have that $\psi \geq -1$, $dP \times n_p(t, dx) \times dt$-a.s. If Q is equivalent to P, then, by the Lenglart-Girsanov theorem, $W_t^Q = W_t - \int_0^t q_s ds$ is a Brownian motion and the process $\tilde{N}$ has compensator $n^Q(s, dx) := (1 + \psi_s(x))n_p(dx)$ under Q (see, for instance, Jacod and Shiryaev [49], Ch. 3, Th. 3.11). Consequently, $1 + \psi$ is the new density of the jump component under Q. Hence, q may be seen as an additional drift that the reference model P may have failed to detect, and ψ may be seen as a misspecification of the size and frequency of the jumps under P. (The model P corresponds to $q = \psi = 0$.)

Next, let us discuss which class of plausibility indices (penalty functions) to use in (2.2)-(2.3). For dynamic choice under uncertainty, the notion of time-consistency plays an important role. A dynamic evaluation $(U_t(F))_{0 \le t \le T}$ is time-consistent if, for $t \ge s$,

$$U_t(F_2) \ge U_t(F_1) \quad \text{implies} \quad U_s(F_2) \ge U_s(F_1).$$

In other words, if F_2 is preferred over F_1 under all possible scenarios at some time t, then F_1 should also have been preferred before time t; see the references provided in the Introduction for further details.

The next theorem shows that a dynamic evaluation $U_t(F) = \inf_{\{Q \sim P \text{ on } \mathcal{F}_t\}}\{\mathrm{E}_Q[F|\mathcal{F}_t] + c_t(Q)\}$ induces time-consistent decisions (or, equivalently with linear utility as we show, induces recursiveness or Bellman's principle), if and only if a lower semi-continuous plausibility index c takes a certain form, specified below. The theorem extends a very recent result (restated as (i)-(iii) and (b) in the theorem) by Tang and Wei [84], who in turn generalized to a setting with jumps an earlier result of Delbaen, Peng and Rosazza Gianin [23] obtained in a Brownian setting.

Theorem 3.1 *Suppose that U takes the form $U_t(F) = \operatorname{ess\,inf}_{\{Q \sim P | Q = P \text{ on } \mathcal{F}_t\}}\{\mathrm{E}_Q[F|\mathcal{F}_t] + c_t(Q)\}$ with $c_t(P) = 0$ and that there are only finitely many markers, i.e., there exists $x_1, \ldots, x_k \in \mathbb{R}$ such that $n_p(dx) = a^1\delta_{x_1}(dx) + \ldots + a^k\delta_{x_k}(dx)$ for positive constants $a^1, \ldots, a^k$, with δ_x the Dirac measure. Then the following statements are equivalent:*

(i) U is time-consistent on L^2.

(ii) U is recursive, i.e., U satisfies Bellman's principle, meaning that for every $t \in [0, T]$, $F \in L^2$ and $A \in \mathcal{F}_t$, we have $U_0(U_t(F)I_A) = U_0(FI_A)$.

(iii) There exists a function

$$\begin{array}{llllllll} r: & [0,T] & \times & \Omega & \times & \mathbb{R}^d & \times & L^2(n_p) & \to & \mathbb{R} \cup \{\infty\} \\ & (t, & & \omega, & & q, & & \psi) & \longmapsto & r(t, \omega, q, \psi), \end{array}$$

measurable with respect to $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{U}$ (with $\mathcal{U}$ denoting the Borel σ-algebra on $L^2(n_p)$), which is convex and lower semi-continuous in (q, ψ) such that for every $t \in [0, T]$,

$$c_t(Q) = \mathrm{E}_Q\left[\int_t^T r(s, q_s, \psi_s)ds \Big| \mathcal{F}_t\right]. \tag{3.3}$$

Furthermore, for a general, possibly infinite, measure n_p (not necessarily restricted to only finitely many markers) we have:

(a) (i) ⇔ (ii) and (iii) ⇒ (ii) (still) hold.

(b) (i) (or (ii)) (still) implies that there exists a function $r(t,q,\psi)$ such that (3.3) holds for all $Q \in A := \cup_{n=1}^{\infty} A_n$, where

$$A_n = \Big\{ Q \ll P \Big| |q_t| \le n \text{ and } -\frac{n-1}{n}(1\wedge|x|) \le |\psi_t(x)| \le n(1\wedge|x|), \text{ for } dt\times n_p(dx)\text{-a.s. all } t,x \Big\}.$$

(c) If, for a $p>1$, c has a nonempty relative interior (with respect to the L^p-norm topology) given by

$$\text{ri}(\text{dom}(c)) := \Big\{ Q \in \mathcal{Q} |\, \text{there exists an } \varepsilon > 0 \text{ such that for all } Q' \in \mathcal{Q} \text{ with}$$
$$\Big|\Big| \frac{dQ'}{dP} - \frac{dQ}{dP} \Big|\Big|_{L^p} \le \varepsilon \text{ we have } c(Q') < \infty \Big\},$$

and if c is continuous on ri(dom(c)), then, under (i) (or (ii)), (3.3) must (even) hold for all $Q \in \{Q \ll P | \psi_t \in L^2(n_p)$ with $\psi_t(x) \ge -(1\wedge|x|)$, for $dt \times n_p(dx)$-a.s. all $t,x\}$.

Remark 3.2 In the case of multiple priors, i.e., $U_t(F) = \operatorname{ess\,inf}_{Q\in M} \mathrm{E}_Q\left[u(F)|\mathcal{F}_t\right]$, with an arbitrary strictly increasing utility function u, one may, for optimization purposes, also consider the corresponding certainty equivalent $\mathrm{CE}_t(F) := u^{-1}\big(\operatorname{ess\,inf}_{Q\in M} \mathrm{E}_Q\left[u(F)|\mathcal{F}_t\right]\big)$; it induces the same optima. As CE_t is invariant under linear transformations of u, one may assume, without loss of generality, that $u(0)=0$. Now (CE_t) satisfies Bellman's principle if and only if the evaluation defined by $\hat{U}_t(F) := \operatorname{ess\,inf}_{Q\in M} \mathrm{E}_Q\left[F|\mathcal{F}_t\right]$ does. If $\hat{U}_t(F)$ is recursive, then, for any $t\in[0,T]$ and $A \in \mathcal{F}_t$, we have

$$\begin{aligned}\mathrm{CE}_0(\mathrm{CE}_t(F)I_A) &= u^{-1}(\hat{U}_0(u(u^{-1}(\hat{U}_t(u(F)))I_A))) = u^{-1}(\hat{U}_0(u(u^{-1}(\hat{U}_t(u(FI_A))))))\\ &= u^{-1}(\hat{U}_0(\hat{U}_t(u(FI_A)))) = u^{-1}(\hat{U}_0(u(FI_A))) = \mathrm{CE}_0(FI_A),\end{aligned} \tag{3.4}$$

where we used in the second equality that $u^{-1}(0)=u(0)=0$. From (3.4), it follows immediately that, for $s \le t$, we have that $\mathrm{CE}_t(F_2) \ge \mathrm{CE}_t(F_1)$ implies that

$$\mathrm{CE}_s(F_2) = \mathrm{CE}_s(\mathrm{CE}_t(F_2)) \ge \mathrm{CE}_s(\mathrm{CE}_t(F_1)) = \mathrm{CE}_s(F_1).$$

Therefore, in the case of multiple priors, requiring recursiveness under an arbitrary utility function can be reduced to requiring recursiveness under linear utility, so that Theorem 3.1 can be applied. Notice that r in this case is equal to an indicator function (in the sense of convex analysis). Therefore, under multiple priors, (3.3) corresponds to the existence of a convex, closed, set-valued predictable mapping, say C, taking values in $\mathbb{R}^d \times L^2(n_p)$ such that $r(s,q,\psi) = I_{C_s}(q,\psi)$.

Remark 3.3 We have seen in Remark 3.2 that multiple priors is time-consistent if one considers terminal payoffs F. However, if one considers (discounted) *payment streams*, then Geman and Ohana [36] show that already expected utility can induce time-inconsistent preferences. Time-consistency in their (discrete-time) setting can be defined by postulating that if, in every scenario at time $t+1$, a payment stream $(A_s)_{s=t+1,\dots,T}$ is preferred over a payment stream $(B_s)_{s=t+1,\dots,T}$ and additionally at time t the payment resulting from A, say A_t, is larger than the payment resulting from B, B_t, then, at time t, $(A_s)_{s=t,\dots,T}$ should also be preferred over $(B_s)_{s=t,\dots,T}$; see Definition 2.1 in Geman and Ohana [36]. Geman and Ohana show that the evaluation defined as the certainty equivalent under expected utility of the discounted sum of the payment stream (i.e., $U_t(A) = u^{-1}\Big(\mathrm{E}_Q\left[u(\sum_{s=t}^T \beta^{s-t} A_s)|\mathcal{F}_t\right]\Big)$, with β a discount factor) is not time-consistent. By summing up payment streams over time, preferences at different points in time are not necessarily preserved, due to changes in wealth. For example, suppose for simplicity that $\beta = 1$ and that at every scenario at time $t+1$, $(A_s)_{s=t+1,\dots,T}$ is preferred over $(B_s)_{s=t+1,\dots,T}$ (e.g., because B entails the possibility of a large future loss). Suppose furthermore that A and B generate the same positive, degenerate payoff at time t, say $m > 0$, so $A_t = B_t = m$. Then, by adding m to the discounted sum of both future payment streams, i.e., by considering at time t the payoffs $m + \sum_{s=t+1}^T A_s$ and $m + \sum_{s=t+1}^T B_s$, the agent can now be less risk averse compared to time $t+1$, because his wealth has changed. Therefore, at time t, he may prefer the payment stream $(B_s)_{s=t,\dots,T}$ over $(A_s)_{s=t,\dots,T}$ (although at time t both payment streams pay the same amount, m, and at $t+1$, A is always preferred over B). This issue does not arise, however, when only considering *terminal* payoffs, as we do (or when considering additive expected utility).

Note that in the absence of time-consistency, the agent would today consciously choose portfolio strategies that he knows he will regret in every future scenario. Theorem 3.1 shows that time-consistency automatically induces (is essentially equivalent to) a structure as in (3.3). Conversely, penalty functions given by (3.3) always lead to time-consistent preferences. Hence, we will postulate that c has the structure given by (3.3). That is, we henceforth assume:

(H1) (c_t) is of the form

$$c_t(Q) = \mathrm{E}_Q\left[\int_t^T r(s, q_s, \psi_s) ds \Big| \mathcal{F}_t\right], \text{ for all } Q \ll P,$$

for a suitably measurable function $r : [0,T] \times \Omega \times \mathbb{R}^d \times L^0(n_p) \to \mathbb{R}_0^+ \cup \{\infty\}$. Furthermore, we assume that r is convex in (q, ψ) and that $r(t, 0, 0) = 0$.

Note that, since r is non-negative, r is minimal at $q = 0$ and $\psi = 0$. These values of q and ψ correspond to the probabilistic model P. Hence, the reference model has the highest esteemed plausibility. At first sight, $L^0(n_p)$ may seem a rather large space for ψ. But condition (H2) below will restrict the values of ψ considered somewhat, and will set r equal to infinity for implausible, not well-integrable, ψ.

Let us consider some examples for which (H1) is true:

Examples 3.4 (1) Relative Entropy: A standard example of the plausibility index in (2.2) is the relative entropy (Csiszár [20], Ben-Tal [7]) defined as

$$c_t(Q) = \alpha H_t(Q|P), \quad \alpha > 0, \quad \text{with} \quad H_t(Q|P) = \begin{cases} \mathrm{E}_Q\left[\log\left(\frac{dQ}{dP}\right)\Big|\mathcal{F}_t\right], & \text{if } Q \in \mathcal{Q}; \\ \infty, \text{ otherwise.} \end{cases}$$

The relative entropy is also known as the Kullback-Leibler divergence; it measures the distance between the distributions Q and P. The relative entropy is used e.g., by Hansen and Sargent [42, 43] in the context of model robustness in macroeconomics. The interpretation is that the economic agent has a reference measure P, but the measure P is merely an approximation to the probabilistic model rather than the true model. As such, the agent does not fully trust the measure P and considers many measures Q, with esteemed plausibility decreasing proportionally to their distance from the approximation P. The parameter α may be viewed as measuring the degree of trust the agent puts in the reference measure P, with the limiting case $\alpha \uparrow \infty$ (respectively, $\alpha \downarrow 0$) corresponding to a maximal degree of trust (respectively, distrust).

In our setting, it may be seen that

$$c_t(Q) = \alpha H_t(Q|P) = \mathrm{E}_Q \left[\int_t^T r(s, q_s, \psi_s) ds \middle| \mathcal{F}_t \right], \tag{3.5}$$

with q and ψ corresponding to the measure Q according to (3.1), and $r(s, q, \psi) := \frac{\alpha}{2}|q|^2 + \int_{\mathbb{R}\setminus\{0\}} \alpha \Psi(\psi(x)) n_p(dx)$ with

$$\Psi(y) = \begin{cases} (1+y)\log(1+y) - y, & \text{if } y \geq -1; \\ \infty, & \text{otherwise}; \end{cases}$$

see Proposition A.25 in the Appendix.

(2) Known Mean: Let us consider the case in which the agent knows the mean return (which, because interest rates are normalized to zero, corresponds to knowing the excess return), (b_t). In this case, the agent restricts attention to probabilistic models of the stochastic evolution of S with mean return equal to (b_t). By the Girsanov-Lenglart theorem, the mean return (b_t^Q) of S under a measure Q is given by

$$b_t^Q = b_t + \sigma_t q_t + \psi_t \cdot \beta_t.$$

Therefore, the agent will only consider probabilistic models that lie in the set

$$M = \left\{ Q \ll P | (b_t^Q) = (b_t) \right\} = \left\{ Q \ll P | \sigma_t q_t + \psi_t \cdot \beta_t = 0, \quad \text{for Lebesgue-a.s. all } t \right\}.$$

This corresponds to a penalty function $c_t(Q) = \mathrm{E}_Q \left[\int_t^T r(s, q_s, \psi_s) ds \middle| \mathcal{F}_t \right]$, with

$$r(s, q, \psi) = \begin{cases} 0, & \text{if} \quad \sigma_s q + \psi \cdot \beta_s = 0; \\ \infty, & \text{otherwise}. \end{cases}$$

In particular, the evaluation satisfies (H1). Note that if $b = 0$ and u is linear, U corresponds to the lower no-arbitrage bound (which is equal to minus the superhedging price of minus the payoff).

However, notice that, in full generality, the set of probabilistic models, M, is non-compact, which may lead to an ill-posed portfolio choice problem. For example, if we have a terminal payoff F, independent of S, then the corresponding evaluation would be given by an essential infimum leading to a degenerate, non-semimartingale evaluation. Therefore, we will assume that the agent only considers additional drifts and additional jump densities below a certain bound, i.e., he only considers additional drifts q satisfying $|q| \leq B$ for a constant $B > 0$ and additional jump densities ψ satisfying $d^- \leq \psi \leq d^+$ with boundary functionals $d^{\pm} \in L^2(n_p)$ and $-1+\epsilon \leq d^- \leq 0 \leq d^+$ for an $\epsilon > 0$. This yields a penalty function of the form

$$r(s,q,\psi) = \begin{cases} 0, & \text{if} \quad \sigma_s q + \psi \cdot \beta_s = 0, \quad |q| \leq B, \quad d^- \leq \psi \leq d^+; \\ \infty, & \text{otherwise.} \end{cases}$$

Of course, B and $d^{\pm}$ could also be time-dependent.

(3) Interval Mean: Now let us consider the case in which the agent is certain that the mean return, (b_t), lies between known lower and upper bounds, (b_t^-) and (b_t^+), respectively. In this case, the agent restricts attention to probabilistic models that give rise to a stochastic evolution of S with mean (b_t) taking values between (b_t^-) and (b_t^+). As before, we will assume that the agent considers only additional drifts q satisfying $|q| \leq B$ for a constant $B > 0$ and additional jump densities ψ satisfying $d^- \leq \psi \leq d^+$ with $d^{\pm} \in L^2(n_p)$ and $-1+\epsilon \leq d^- \leq 0 \leq d^+$ for an $\epsilon > 0$. Hence, the agent only considers probabilistic models that lie in the set given by

$$\begin{aligned} M &= \Big\{Q \ll P | b_t^- \leq b_t^Q \leq b_t^+, \quad |q_t| \leq B, \quad d^- \leq \psi_t \leq d^+, \quad \text{for Lebesgue-a.s. all } t\} \\ &= \Big\{Q \ll P | b_t^- - b_t \leq \sigma_t q_t + \psi_t \cdot \beta_t \leq b_t^+ - b_t, \quad |q_t| \leq B, \\ &\qquad d^- \leq \psi_t \leq d^+, \quad \text{for Lebesgue-a.s. all } t\Big\}. \end{aligned}$$

This corresponds to a penalty function $c_t(Q) = \mathrm{E}_Q\left[\int_t^T r(s,q_s,\psi_s)ds \Big| \mathcal{F}_t\right]$, with

$$r(s,q,\psi) = \begin{cases} 0, & \text{if} \quad b_s^- - b_s \leq \sigma_s q + \psi \cdot \beta_s \leq b_s^+ - b_s, \quad |q| \leq B, \text{ and } d^- \leq \psi \leq d^+; \\ \infty, & \text{otherwise.} \end{cases}$$

(4) Discrete dynamic worst case scenarios: Suppose that, at each future time instance $s > t$, the agent considers a family of finitely many values $q_{1,s}, \ldots, q_{L,s}$ for the future drift, q_s, and finitely many values $\psi_{1,s}, \ldots, \psi_{L,s}$ for the future jump density, ψ_s, all equally plausible. He then decides to adopt a worst case approach by taking the expectation with respect to each corresponding measure and next computing the minimum. That is, let

$$M = \Big\{Q \in \mathcal{Q} \Big| \text{ for Lebesgue-a.s. all } s: \ (q_s, \psi_s) \in \{(q_{i,s}, \psi_{j,s}) |\, i,j \in \{1,\ldots,L\}\}\Big\}.$$

This corresponds to a penalty function

$$c_t(Q) = \mathrm{E}_Q\left[\int_t^T r(s,q_s,\psi_s)ds \Big| \mathcal{F}_t\right], \tag{3.6}$$

where r takes the form

$$r(s,q,\psi) = \begin{cases} 0, & \text{if} \quad (q,\psi) \in \mathrm{conv}\Big(\{(q_{i,s},\psi_{j,s}) | i,j \in \{1,\ldots,L\}\}\Big); \\ \infty, & \text{otherwise.} \end{cases}$$

(Recall that conv$(\cdot)$ of a set is given by its convex hull.) The reason that the penalty function defined in (3.6) induces the same preferences as the indicator function (in the sense of convex analysis) of M is that the minimum of a convex function taken over a compact convex set can always assumed to be attained in the extreme points of the set.

Notice further that by redefining the reference measure, one may assume, without loss of generality, that $0 \in \text{conv}\Big(\{(q_{i,s}, \psi_{j,s}) | i, j \in \{1, \dots, L\}\}\Big)$.

(5) Ball-Robustification: Suppose the agent wants to test the robustness of the reference measure P. In this case, he restricts attention to alternative probabilistic models Q that are contained in a small ball around P, say $M = \{Q \ll P | |q_t| \le \hat{\delta}, \quad \hat{\epsilon}^- \le \psi_t \le \hat{\epsilon}^+, \text{ for Lebesgue-a.s. all } t\}$, for a $\hat{\delta} > 0$ and deterministic functions $\hat{\epsilon}^{\pm} \in L^2(n_p)$ satisfying $-1 + \epsilon \le \hat{\epsilon}^- \le 0 \le \hat{\epsilon}^+$ for an $\epsilon > 0$. This gives rise to the penalty function $c_t(Q) = \mathrm{E}_Q\left[\int_t^T r(s, q_s, \psi_s) ds \middle| \mathcal{F}_t\right]$, with

$$r(s, q, \psi) = \begin{cases} 0, & \text{if} \quad |q| \le \hat{\delta}, \quad \hat{\epsilon}^- \le \psi \le \hat{\epsilon}^+; \\ \infty, & \text{otherwise.} \end{cases}$$

Because ambiguity with respect to the jump part is reflected by a whole functional, ψ, in the argument of r (rather than a real number), we need proper integrability conditions. Specifically, we need the definition of an Orlicz space. Let $\tilde{\Psi}(x) := (1 + |x|)\log(1 + |x|) - |x|$. Note that $\tilde{\Psi}$ is a Young function, meaning that it is lower semi-continuous, convex, not identical zero, and symmetric with $\tilde{\Psi}(0) = 0$. A slight modification of the standard Birnbaum-Orlicz space is given by the Banach space

$$L^{\tilde{\Psi}}(n_p) := \left\{ f \in L^0(n_p) | \int_{\mathbb{R}\setminus\{0\}} \tilde{\Psi}(af(x)) n_p(dx) < \infty \text{ for some } a > 0 \right\},$$

with Luxemburg norm

$$|f|_{L^{\tilde{\Psi}}(n_p)} := \inf \left\{ a > 0 | \int_{\mathbb{R}\setminus\{0\}} \tilde{\Psi}\Big(\frac{f(x)}{a}\Big) n_p(dx) \le 1 \right\}.$$

Generally, Banach spaces with Luxemburg norms arising from a Young function, say H, are called Orlicz spaces and, as above, are denoted by $L^H(n_p)$. If $H(x) = x^p$ with $p \ge 1$, the corresponding Orlicz space is given by $L^p(n_p)$. For applications of the theory of Orlicz spaces, see, for instance, Hindy, Huang and Kreps [46], Biagini and Frittelli [10], Cheridito and Li [16], and Drapeau and Kupper [27]. See also Rao and Ren [74] for an overview of Orlicz space theory.

To ensure that the problem is well-posed and solvable in terms of BSDEs, we need to assume proper growth and regularity conditions on the penalty function. These assumptions should be as weak as possible to include most penalty functions. In particular, our assumptions should be satisfied for all our examples above. In a recent paper, Delbaen, Hu and Bao [24] showed, in a Brownian setting, that if the penalty function grows slower than the relative entropy, then the corresponding (dual) superquadratic BSDEs we will derive below, do not have a solution. (In this case, one could conceivably still try to approach the problem with supersolutions to BSDEs; see Drapeau, Heyne and Kupper [26]. However, this would make numerical computations highly challenging and also change the nature and interpretation of the solutions.) Therefore, we will only consider penalty functions that can be bounded from below in terms of the relative entropy, i.e., we will assume that there exist $K_1', K_2' > 0$ such that $c(Q) \geq -K_1' + K_2' H(Q|P)$. In view of (3.5), this corresponds to r satisfying the following growth condition, henceforth denoted by (H2):

(H2) There exist constants $K_1, K_2 > 0$ such that for all t, all q, and all $\psi \in L^0(n_p)$,

$$r(t, q, \psi) \quad \geq \quad -K_1 + K_2 \left(|q|^2 + \int_{\mathbb{R}\setminus\{0\}} [(1+\psi_t(x)) \log(1+\psi_t(x)) - \psi_t(x)] n_p(dx) \right).$$

Furthermore, we suppose that r is weak* lower semi-continuous in $(q, \psi) \in \mathbb{R}^d \times L^{\tilde{\Psi}}(n_p)$.

Because there exists a constant $C > 0$ such that $C\Psi \geq \tilde{\Psi}$, assumption (H2) implies that only functions $\psi \in L^{\tilde{\Psi}}(n_p)$ need to be considered. Thus, we will for the remainder of this paper restrict $r(t, \omega, \cdot, \cdot)$ to the Banach space $\mathbb{R}^d \times L^{\tilde{\Psi}}(n_p)$.

The next condition (H3) corresponds to the condition needed by Kobylanski [55] in a Brownian filtration, to guarantee that a comparison principle holds. We extend the condition so that it encompasses our setting with jumps. Specifically, we suppose that (sub)differentials of the penalty functions r can be bounded from below in terms of the differentials of the relative entropy. (The notion of subdifferentiability on a Banach space is a generalization of the usual definition of differentiability; it is frequently used for convex functions, see the Appendix for details.) That is, we henceforth assume:

(H3) There exist constants $\hat{K}_1, \hat{K}_2 > 0$, such that for all (t, q) and $\psi \in L^{\tilde{\Psi}}(n_p)$

$$|\partial_q r(t, q, \psi)| \geq -\hat{K}_1 + \hat{K}_2 |q|. \tag{3.7}$$

Furthermore, there exist $\tilde{K} \in L^2(n_p)$ and $\hat{K}_3 > 0$ such that for all (t, q) and all $\psi \in L^{\tilde{\Psi}}(n_p)$ we have

$$|\partial_\psi r(t, q, \psi)| \geq -\tilde{K} + \hat{K}_3 |\log(1+\psi)|. \tag{3.8}$$

These inequalities should hold for every element of the corresponding subdifferential, where we set $|\emptyset| = \infty$. (Our proofs will show that (3.8) actually only needs to hold for those elements of $\partial_\psi r$ that are bounded by a fixed constant.) We will see later that (H3) entails that the subgradients of the drivers of our BSDEs satisfy conditions that, in the case of no jumps, correspond to the conditions needed by Kobylanski [55] in a Brownian setting. Of course, conditions (H1)-(H3) are satisfied in all our examples above.

4 The Optimization Problems and Their Solutions

We are interested in the following optimization problem:

$$V_t(F) = \max_{\pi \in \mathcal{A}} U_t(F + X_T^{(\pi)}), \tag{4.1}$$

where U_t is defined in (2.3) with plausibility index c satisfying (H1)-(H3), $\mathcal{A}$ is the set of admissible trading strategies, F is the (bounded) payoff at maturity and $X_T^{(\pi)}$ is the wealth process.

4.1 Linear Utility Under Variational Preferences

We first assume that the utility function u in (2.3) is linear. Then one may see from Lemma A.35 in the Appendix that, for every admissible π, $U_t(F + X_T^{(\pi)})$ is finite. We solve problem (4.1) with the help of BSDEs. Define

$$g(t,z,\tilde{z}) := \sup_{q\in\mathbb{R}^d,\psi\in L^{\tilde{\Psi}}(n_p)} \Big\{ zq + \tilde{z}\cdot\psi - r(t,q,\psi) \Big\}, \tag{4.2}$$

for $t \in [0,T]$, $z \in \mathbb{R}^{1\times d}$, and $\tilde{z} \in L^2(n_p(dx)) \cap L^\infty(n_p(dx))$. It follows from Lemma A.16 in the Appendix that g is real-valued, suitably measurable, and $g \geq 0$ with equality if $z = 0$ and $\tilde{z} = 0$. So g assumes its minimum at zero. The next theorem shows that, under our assumptions, $U(F)$ is the unique solution to a BSDE with terminal condition F and driver function g.

Theorem 4.1 *Assume that (H1)-(H3) hold. Then $U_t(F)$ is the unique solution to the BSDE*

$$\begin{aligned} dU_t(F) &= g(t,Z_t,\tilde{Z}_t)dt - Z_t dW_t - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_t(x)\tilde{N}_p(dt,dx), \\ U_T(F) &= F. \end{aligned} \tag{4.3}$$

As a by-product, while proving this theorem, we show in the Appendix that every BSDE with driver function g growing at most quadratically (for $\tilde{z}$ in a uniformly bounded set) has a unique solution satisfying a comparison principle; see Theorem A.29, Remark A.31, and Proposition A.34. This generalizes Kobylanski's theorem on existence and uniqueness of solutions of quadratic BSDEs, one of the main results in the BSDE literature, to an infinite activity jump setting.

If $g = 0$ would hold, the evaluation U would correspond to a conditional expectation; see Example 2.3 of Section 2.2. However, our economic agent is ambiguity averse, considering all alternative probabilistic models, with different degrees of esteemed plausibility. As a result, $g \geq 0$ which decreases the evaluation. Z is the stochastic (Malliavin) derivative of the evaluation with respect to W. Comparing (4.3) with (2.1), we see that Z and $\tilde{Z}$ play the same role for $U(F)$ as σ and β for the instantaneous return of the asset price. Therefore, Z and $\tilde{Z}$ may be seen as measuring the degree of fluctuation ('variability') of the evaluation coming from the Brownian motion and from the jumps, respectively. The larger $|Z|$, the more variability is due to the local Gaussian part, and the larger $|\tilde{Z}|$, the more variability is due to the jump component of the model. Next, for illustration purposes, we may employ the penalty functions of Examples 3.4 and compute the corresponding driver functions using (4.2). This yields the following driver functions:

Examples 4.2 (1) Relative Entropy: As the Fenchel dual conjugate of $\alpha((1+x)\log(1+x) - x)$ is given by $\alpha(e^{x/\alpha} - \frac{x}{\alpha} - 1)$, one verifies that in this case

$$g(t,z,\tilde{z}) = \frac{1}{2\alpha}|z|^2 + \alpha\int_{\mathbb{R}\setminus\{0\}} \left(\exp\left\{\frac{\tilde{z}(x)}{\alpha}\right\} - 1 - \frac{\tilde{z}(x)}{\alpha}\right) n_p(dx).$$

(2) Known Mean: First, note that, as in this case the penalty function only takes the values zero and infinity, $g(t,\cdot,\cdot)$ must be positively homogeneous. Define $\langle(z,\tilde{z}),(q,\psi)\rangle := qz +$

$\tilde{z} \cdot \psi$ and $C_t := \{(q, \psi) \in \mathbb{R}^d \times L^{\tilde{\Psi}}(n_p) : A_t(q, \psi)^\intercal = 0, \ |q| \le B, \ d^- \le \psi \le d^+\} \subset \mathbb{R}^d \times L^2(n_p)$, where the matrix $A_t = (\sigma_t, \beta_t)$ maps from $\mathbb{R}^d \times L^2(n_p)$ to $\mathbb{R}^n$ (with '$\cdot$' denoting multiplication in the second component). Then we have

$$g(t, z, \tilde{z}) = \sup_{(q,\psi)\in C_t} \langle (z, \tilde{z}), (q, \psi) \rangle.$$

Next, note that in the case that the only constraint is $A_t(q, \psi)^\intercal = 0$ (so that $B = \infty$ and $d^\pm = \pm\infty$), the solution to the supremum above would be 0 if $(z, \tilde{z})^\intercal \in \text{Image}(A_t^*)$ and infinity else, where $A_t^* = (\sigma_t, \beta_t)^\intercal$ is the adjoint transpose of A_t. This is due to the fact that if there exists $y \in \mathbb{R}^n$ such that $(z, \tilde{z})^\intercal = A_t^* y$, then

$$\langle (z, \tilde{z}), (q, \psi) \rangle = \langle (A_t^* y)^\intercal, (q, \psi) \rangle = \langle y, A_t(q, \psi)^\intercal \rangle = 0,$$

for $(q, \psi) \in C_t$. Now, given $(z, \tilde{z}) \in \mathbb{R}^d \times (L^2(n_p) \cap L^\infty(n_p))$, we can find a unique orthogonal decomposition

$$(z, \tilde{z}) = P_t(z, \tilde{z}) + (z, \tilde{z})^\perp,$$

where $P_t(z, \tilde{z})$ is the projection of $(z, \tilde{z})$ on the image of A_t^*. Denote the first component of $P_t(z, \tilde{z})$ by $P_t(z)$ and the second component of $P_t(z, \tilde{z})$ by $P_t(\tilde{z})$. Define $z^\perp$ and $\tilde{z}^\perp$ similarly. Then we obtain

$$\begin{aligned} g(t, z, \tilde{z}) &= \sup_{(q,\psi)\in C_t} \Big\{ P_t(z)q + P_t(\tilde{z}) \cdot \psi + z^\perp q + \tilde{z}^\perp \cdot \psi \Big\} \\ &= \sup_{(q,\psi)\in C_t} \{ z^\perp q + \tilde{z}^\perp \cdot \psi \} = B|z^\perp| + d^+ \cdot \tilde{z}^{+,\perp} - d^- \cdot \tilde{z}^{-,\perp}, \end{aligned}$$

where, for all x, we define $\tilde{z}^{+,\perp}(x) = \tilde{z}^\perp(x)$ if $\tilde{z}^\perp(x) > 0$ and zero else; $\tilde{z}^{-,\perp}$ is defined similarly.

(3) Interval Mean: As in this case only probabilistic models are considered that yield an excess return within a certain predefined interval, we get from (4.2) that

$$g(t, z, \tilde{z}) = \sup_{(q,\psi)\in C_t} \{ zq + \tilde{z} \cdot \psi \}, \tag{4.4}$$

where

$$C_t = \Big\{ (q, \psi) \in \mathbb{R}^d \times L^{\tilde{\Psi}}(n_p) | b_t^- - b_t \le \sigma_t q + \psi \cdot \beta_t \le b_t^+ - b_t, \quad |q| \le B, \quad d^- \le \psi \le d^+ \Big\}.$$

In full generality, it is not possible to simplify the driver function further. However, (4.4) yields that, for fixed $(t, z, \tilde{z})$, the driver function can be obtained as the maximum of a linear programming problem.

(4) Discrete dynamic worst case scenarios: It is straightforward to verify that in the case of a worst case scenario evaluation we obtain

$$g(t, z, \tilde{z}) = \max_{i=1,\ldots,L} q_{i,t} z + \max_{i=1,\ldots,L} \tilde{z} \cdot \psi_{i,t}.$$

(5) Ball-Robustification: In the case of a ball-robustification procedure we get

$$g(t,z,\tilde{z}) = \hat{\delta}|z| + \tilde{z}^{+} \cdot \hat{\epsilon}^{+} - \tilde{z}^{-} \cdot \hat{\epsilon}^{-},$$

concluding our examples.

Next, let

$$f(s,z,\tilde{z}) := \inf_{\pi\in\Pi} \left\{-\pi b_s + g(s, z - \pi\sigma_s, \tilde{z} - \pi\beta_s)\right\}. \tag{4.5}$$

Consider the BSDE

$$Y_t = F - \int_t^T f(s, Z_s, \tilde{Z}_s) ds + \int_t^T Z_s dW_s + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x) \tilde{N}_p(ds,dx), \quad t \in [0,T]. \tag{4.6}$$

Theorem 4.3 *Assume that (H1)-(H3) hold. Then the BSDE (4.6) has a unique solution Y, and $V_0(F) = Y_0 + w_0$. Furthermore, the optimal strategy π^*_s is a predictable process that attains the infimum in (4.5) for $(z,\tilde{z}) = (Z_s, \tilde{Z}_s)$, i.e.,*

$$f(s, Z_s, \tilde{Z}_s) = -\pi^*_s b_s + g(s, Z_s - \pi^*_s\sigma_s, \tilde{Z}_s - \pi^*_s\beta_s).$$

Remark 4.4 The dynamic evaluation under the optimal portfolio choice problem (4.1) is given by $V_t(F) = Y_t + X_t^{(\pi^*)}$. Note that, thanks to the time-consistency of $(U_t(F))_{0\le t\le T}$, the optimal portfolio strategy $(\pi^*_t)_{0\le t\le T}$ is 'time-consistent' as well, in the sense that at every (future) time instance t, the corresponding argmax also agrees with $(\pi^*_s)_{s\ge t}$. If the evaluation were not time-consistent, the agent would at time zero choose π^*_0 *because* it lies on the optimal decision path $(\pi^*_t)_{0\le t\le T}$, although he already knows that he may not stick to this decision path in the future.

Heuristically, the optimal portfolio choice proceeds as follows: The excess return, b, is typically positive. Hence, the term $-\pi b_s$ in the minimization problem (4.5) will 'tempt' the economic agent to invest in risky assets (that is, to pick a positive π) so as to benefit from the excess return. The agent is, however, ambiguity averse. Therefore $(Z, \tilde{Z})$, representing the variability of the evaluation due to the Brownian component and the jumps, is penalized by $g(s, Z_s, \tilde{Z}_s)$ (before hedging). The agent chooses a π to partially hedge Z and $\tilde{Z}$. The aggregate penalty after hedging is given by $g(s, Z_s - \pi_s\sigma_s, \tilde{Z}_s - \pi_s\beta_s)$. Summarizing, when the agent chooses a $\pi \in \Pi$, he faces a tradeoff between:

(a) Benefitting from the excess return $\pi_s b_s$.

(b) Diminishing the variability of the evaluation due to the locally Gaussian part. (That is, choosing π such that $|Z_s - \pi_s\sigma_s|$ is small.)

(c) Diminishing the variability of the evaluation due to jumps. (That is, choosing π such that $|\tilde{Z}_s - \pi_s\beta_s|$ is small.)

Note that if there are only finitely many jumps, (4.5) is a finite dimensional convex optimization problem that can be computed numerically efficiently; see the examples in Section 6 below. Since the portfolio choice problem is translation invariant, it is straightforward to see that the indifference valuation is given by $V_0(F) - V_0(0)$.

4.2 Exponential Utility Under Multiple Priors Preferences

A utility function that is particularly popular in insurance and financial mathematics (Goovaerts et al. [39, 40], Föllmer and Schied [33] and Mania and Schweizer [63]) and decision theory (Gollier [38]) is the exponential utility function. When u is exponential, we provide a solution to the portfolio choice and indifference valuation problems in the case that the penalty function c in (2.3) is an indicator function. This means that we are in the multiple priors setting of Gilboa and Schmeidler [37]. For exponential utility this restriction is necessary because otherwise time-consistency may not hold.

Specifically, let $\gamma > 0$ and consider the robust expected utility optimization problem

$$\begin{aligned} V_0(F) &= \sup_{\pi\in\mathcal{A}} U(F + X_T^{(\pi)}) \\ &:= \sup_{\pi\in\mathcal{A}} \inf_{Q\in M} -\left(\mathrm{E}_Q\left[\exp\left\{-\frac{F + X_T^{(\pi)}}{\gamma}\right\}\right]\right), \end{aligned} \tag{4.7}$$

for a weakly compact set $M \subset \mathcal{Q}$. In order to have time-consistent preferences we assume that M is given by

$$M = \left\{Q \in \mathcal{Q}\,\middle|\, (q_s, \psi_s) \in C_s,\ \text{for every } s \in [0, T]\right\},$$

for a convex, closed, set-valued predictable mapping C taking values in $\mathbb{R}^d \times L^{\tilde{\Psi}}(n_p)$. Furthermore, we assume that C is 'bounded' in the sense that there exist constants $B, \epsilon > 0$ and bounded $L^{2,\infty}$ functions $-1 + \epsilon \le d^- \le 0 \le d^+$ such that for every s and ω we have that $(q, \psi) \in C_s(\omega)$ implies that $|q| \le B$ and $d_s^- \le \psi \le d_s^+$. For examples of the set M, see Examples 3.4: (2)-(5). The case that $C = \{0\}$ corresponds to ambiguity neutrality, i.e., to effectively not considering any alternative probabilistic model at all; in this case, $M = \{P\}$ would hold. As $\exp\{-x\} \ge 0$, the expectation is well-defined for every trading strategy.

Contrary to Section 4.1, in which the economic agent is ambiguity averse but not risk averse (linear utility), the economic agent solving (4.7) is both ambiguity averse and risk averse. Note that $\gamma > 0$ measures the absolute risk tolerance of the agent (with large values of γ corresponding to a low level of risk aversion and low values corresponding to a high level of risk aversion).

Clearly, by construction $C_t(\omega)$ is weakly compact so that we can define

$$\bar{g}(t, z, \tilde{z}) := \sup_{(q,\psi)\in C_t} \left\{zq + \gamma\Big(\exp\left\{\frac{\tilde{z}}{\gamma}\right\} - 1\Big) \cdot \psi\right\} \tag{4.8}$$

$$= zq_t^* + \gamma\Big(\exp\left\{\frac{\tilde{z}}{\gamma}\right\} - 1\Big) \cdot \psi_t^*. \tag{4.9}$$

In case that the set M is defined as in Examples 3.4: (2)-(5), the function $\bar{g}$ is obtained by replacing in the driver functions given in Examples 4.2: (2)-(5), $\tilde{z}$ by $\gamma(e^{\tilde{z}/\gamma} - 1)$.

The next theorem shows that the solution to the optimization problem (4.7) can be obtained directly from the solution to a BSDE:

Theorem 4.5 *The solution to (4.7) is given by* $V_0(F) = -\exp\{-\frac{1}{\gamma}(w_0 + Y_0)\}$*, where* (Y_t) *is the unique solution to the BSDE with terminal condition* F *and driver function:*

$$f(s,z,\tilde{z}) := \inf_{\pi\in\Pi}\Bigg\{ -\pi b_s + \frac{1}{2\gamma}|z-\pi\sigma_s|^2 + \bar{g}(s, z-\pi\sigma_s, \tilde{z}-\pi\beta_s) \\ + \gamma\int_{\mathbb{R}\setminus\{0\}}\left(\exp\left\{\frac{\tilde{z}(x)-\pi\beta_s(x)}{\gamma}\right\} - 1 - \frac{\tilde{z}(x)-\pi\beta_s(x)}{\gamma}\right) n_p(dx)\Bigg\}, \quad (4.10)$$

with $\bar{g}$ *defined in (4.8). Furthermore, the optimal strategy* π^*_s *is a predictable process that attains the infimum in (4.10) for* $(z,\tilde{z}) = (Z_s, \tilde{Z}_s)$*.*

Remark 4.6 If the expectation in (4.7) is replaced by an expectation conditional on $\mathcal{F}_t$, then the optimal solution is given by $V_t(F) = -\exp\{-\frac{1}{\gamma}(X_t^{(\pi^*)} + Y_t)\}$.

Remark 4.7 BSDEs have been a rather popular tool to solve the utility maximization problem under exponential utility, in a wide variety of settings (the exponential utility maximization problem is connected to the popular minimal entropy martingale measure and to the Esscher density). In the case that there are no jumps, i.e., $n_p = 0$, and there is no ambiguity, i.e., $d^+ = d^- = 0$, our general solution above reduces to the solution obtained by Hu, Imkeller and Müller [47]; see also El Karoui and Rouge [30]. These results have been generalized for continuous price processes to continuous and non-continuous filtrations, see for instance, Mania and Schweizer [63] and Becherer [5], in a purely risk-based setting. Recently, Morlais [68] generalized the results by Becherer [5] by allowing for infinite activity jumps in the asset price processes. However, none of these works allow for ambiguity, as opposed to our setting. In the case that there are no jumps but there is (Brownian) ambiguity (i.e., $n_p = d^+ = d^- = 0$) and the trading constraints are assumed to be convex, our general solution above reduces to the solution obtained by Müller [69].

In (4.7), the economic agent is 'penalized' on the one hand for the risk he faces (represented by the γ-exponential utility) and on the other hand for the ambiguity he encounters (represented by the set C).

Note that by (4.9), $\bar{g}(t, Z_t, \tilde{Z}_t)$ can be decomposed into two parts — the first one due to the uncertainty about the Brownian motion and the second one due to the uncertainty in the jump part. Thus, the penalty in (4.10) in total features four terms (the terms with plus sign with two terms due to $\bar{g}$):

(1.) The first term is due to the (local) risk coming from the Brownian motion. This term would equal zero if the agent is not risk averse (i.e., if $\gamma \uparrow \infty$) or if, after hedging, there is no locally Gaussian randomness affecting the evaluation (i.e., if $Z_s - \pi_s\sigma_s = 0$).

(2.) The second term, $\bar{g}(s, Z_s - \pi_s\sigma_s, \tilde{Z}_s - \pi_s\beta_s)$, is due to the (local) model uncertainty. This part again consists of two terms, see (4.9). One part due to the (local) model uncertainty about the Brownian motion and one part due to the (local) model uncertainty about the jumps. The first part would equal zero if there is no model uncertainty in the Brownian part (i.e., if C contains only elements with q equal to zero) or if, after hedging, there is no randomness due to the Brownian part affecting the evaluation (i.e., if $Z_s - \pi_s\sigma_s = 0$). The second part would equal zero if there is no model uncertainty in the jump part (i.e., if C contains only elements with ψ equal to zero) or if, after hedging, there is no randomness due to the jump part affecting the evaluation (i.e., if $\tilde{Z}_s - \pi_s\beta_s = 0$).

(3.) The last term in (4.10) (with plus sign) is due to the (local) risk coming from the jump part; it is the jump analog of (1.).

Note that C, the set of alternative models for the drift q, is uniformly bounded. Consequently, we must have in (4.9) that $|zq^*| \leq \text{const}|z|$. Therefore, in (4.10), the economic agent is penalized quadratically, by $|Z_s - \pi_s\sigma_s|^2$, due to the (local) risk coming from the Brownian motion, and linearly by a penalty bounded by $|Z_s - \pi_s\sigma_s|$, due to the (local) model uncertainty about the Brownian motion. Consequently, if $|Z_s - \pi_s\sigma_s|$ is small, i.e., if there is 'little' Brownian randomness left after hedging, then the penalty due to ambiguity will be larger than the penalty due to risk. From a Taylor expansion of the third term (with plus sign) in (4.10) and the jump part in (4.9), it may be seen that the same observation is true for the jump part. Therefore, if there is only 'little' randomness left after hedging, the evaluation is more (negatively) affected by ambiguity than by risk. On the other hand, if there is 'much' Brownian randomness left after hedging, (meaning that $|Z_s - \pi_s\sigma_s|$ is large), then the penalty due to risk is of a higher order than the penalty due to ambiguity. It is interesting to note, however, that the latter effect is not true for the jump part, since the penalties for risk and ambiguity arising from the jump part are of the same order if $|\tilde{Z}_s - \pi_s\beta_s|$ is large.

Since the problem is again translation invariant, it is straightforward to see that the indifference valuation is given by $-\gamma\log(-V_0(F)) - (-\gamma\log(-V_0(0)))$.

4.3 Logarithmic Utility Under Variational Preferences

We now consider predictable trading strategies ρ that represent the part of wealth (rather than the absolute amount) invested in stock i. The admissible trading strategies are supposed to take values in a compact set $\tilde{C} \subset \mathbb{R}^{1\times n}$. We assume that $\tilde{C}\beta \in [-1+\bar{\delta}, \infty)$ for a $\bar{\delta} > 0$. We denote the set of all admissible trading strategies by $\mathcal{A}$; it is the set of all $\mathbb{R}^{1\times n}$-valued predictable processes ρ with $\rho_t \in \tilde{C}$, $dP \times dt$ a.s. The wealth process $X^{(\rho)}$ of a trading strategy ρ with initial capital w_0 satisfies

$$\begin{aligned} X_t^{(\rho)} &= w_0 + \sum_{i=1}^n \int_0^t X_{u-}^{(\rho)} \frac{\rho_u^i}{S_{u-}^i} dS_u^i \\ &= w_0 + \int_0^t X_{u-}^{(\rho)} \rho_u(\sigma_u dW_u + b_u du) + \int_0^t \int_{\mathbb{R}\setminus\{0\}} X_{u-}^{(\rho)} \rho_u \beta_u(x) \tilde{N}_p(dx, du). \end{aligned} \tag{4.11}$$

It follows that

$$X_t^{(\rho)} = w_0 \mathcal{E}\left(\int_0^t \rho_u\sigma_u dW_u + \int_0^t \rho_u b_u du + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \rho_u\beta_u(x)\tilde{N}_p(du, dx)\right). \tag{4.12}$$

We solve the optimization problem

$$V_0 = \sup_{\rho\in\mathcal{A}} \inf_{Q\in\mathcal{Q}} \mathrm{E}_Q\left[\gamma\log\left(X_T^{(\rho)}\right) + \int_0^T r(s, q_s, \psi_s)ds\right], \quad \gamma > 0. \tag{4.13}$$

Let

$$\begin{aligned} f(s, z, \tilde{z}) := \inf_{\rho\in\tilde{C}} \Big\{ &-\gamma\rho b_s - \gamma\int_{\mathbb{R}\setminus\{0\}} \big[\log(1+\rho\beta_s(x)) - \rho\beta_s(x)\big] n_p(dx) \\ &+ \frac{\gamma}{2}|\rho\sigma_s|^2 + g(s, z-\gamma\rho\sigma_s, \tilde{z}-\gamma\log(1+\rho\beta_s))\Big\}. \end{aligned} \tag{4.14}$$

For examples of r and the corresponding g, see Examples 3.4 and 4.2. For simplicity we assume that f is convex. We consider the BSDE

$$Y_t = 0 - \int_t^T f(s, Z_s, \tilde{Z}_s)ds + \int_t^T Z_s dW_s + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x)\tilde{N}_p(ds, dx), \quad t \in [0, T]. \tag{4.15}$$

Theorem 4.8 *The BSDE (4.15) has a unique solution* $(Y, Z, \tilde{Z})$ *and the solution to (4.13) is given by*

$$V_0 = Y_0 + \gamma \log(w_0).$$

Furthermore, the optimal strategy ρ_s^* *is a predictable process that attains the infimum in (4.14) for* $(z, \tilde{z}) = (Z_s, \tilde{Z}_s)$.

Remark 4.9 If the expectation in (4.13) is replaced by an expectation conditional on $\mathcal{F}_t$, then the optimal dynamic evaluation for a logarithmic utility function is given by $V_t = Y_t + \gamma \log(X_t^{(\pi^*)})$.

Remark 4.10 In the case that there are no jumps and there is no ambiguity, problem (4.13) is solved by Hu, Imkeller and Müller [47]. The case of ambiguity without jumps is considered by Müller [69]. For the case of a degenerate jump distribution with a penalty function c given by the relative entropy, see Jeanblanc, Matoussi and Ngoupeyou [50]. These results all occur as special cases of our general solution provided above.

4.4 Power Utility Under Multiple Priors Preferences

When u is power, we provide a solution to the portfolio choice problem in the case that the penalty function c in (2.3) is an indicator function. This means that we are in the multiple priors setting of Gilboa and Schmeidler [37]. For a power utility function this restriction is necessary because otherwise time-consistency may not hold.

As in the logarithmic utility case, we consider predictable trading strategies ρ that represent the part of wealth invested in stock i taking values in a convex and compact set $\tilde{C} \subset \mathbb{R}^{1\times n}$ with $\tilde{C}\beta \in [-1 + \bar{\delta}, \infty)$ for a $\bar{\delta} > 0$. The robust expected utility optimization problem with a power utility function then takes the form

$$V_0 \quad = \quad \sup_{\rho \in \mathcal{A}} \inf_{Q \in M} \mathrm{E}_Q \left[\frac{(X_T^{(\rho)})^\gamma}{\gamma} \right], \tag{4.16}$$

where we assume that $\gamma < 0$ and (in order to induce time-consistent decisions) that the set $M \subset \mathcal{Q}$ has the same form as in the exponential case. Let

$$\begin{aligned} f(s, z, \tilde{z}) := \inf_{\rho \in \tilde{C}} \Bigg\{ & \gamma \rho b_s + \frac{\gamma(\gamma - 1)}{2} |\rho \sigma_s|^2 + \frac{|z|^2}{2} + \gamma \rho \sigma_s z + \bar{g}(s, \gamma \rho \sigma_s + z, \gamma \log(1 + \rho \beta_s) + \tilde{z}) \\ & + \int_{\mathbb{R}\setminus\{0\}} \Big[(1 + \rho \beta_s(x))^\gamma e^{\tilde{z}(x)} - \tilde{z}(x) - \gamma \rho \beta_s(x) - 1 \Big] n_p(dx) \Bigg\}, \end{aligned} \tag{4.17}$$

where the function $\bar{g}$ was defined in (4.8). We consider the BSDE

$$Y_t = 0 - \int_t^T f(s, Z_s, \tilde{Z}_s)ds + \int_t^T Z_s dW_s + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x)\tilde{N}_p(ds, dx), \quad t \in [0, T]. \tag{4.18}$$

Theorem 4.11 *The BSDE (4.18) has a unique solution $(Y, Z, \tilde{Z})$ and the solution to (4.16) is given by*

$$V_0 = \frac{w_0^\gamma}{\gamma} \exp\{-Y_0\}.$$

Furthermore, the optimal strategy ρ_s^* is a predictable process that attains the infimum in (4.17) for $(z, \tilde{z}) = (Z_s, \tilde{Z}_s)$.

Remark 4.12 For power utility, the dynamic evaluation under the optimal portfolio choice is given by $V_t = \frac{(X_t^{(\pi^*)})^\gamma}{\gamma} \exp\{-Y_t\}$.

Remark 4.13 In the case that there are no jumps and there is no ambiguity, problem (4.16) has been considered by Hu, Imkeller and Müller [47]. The case of ambiguity without jumps has been considered by Müller [69]. To the best of our knowledge, the case with jumps, whether finite or infinite jump activity and whether with or without ambiguity, has not been solved as yet.

5 Decomposition of the Excess Return

Let us consider the case in which the trading set Π is specified as $[\pi^1_{\text{lower}}, \pi^1_{\text{upper}}] \times \ldots \times [\pi^n_{\text{lower}}, \pi^n_{\text{upper}}]$, for $-\infty < \pi^i_{\text{lower}} \le \pi^i_{\text{upper}} < \infty$, $i = 1, \ldots, n$. That is, the economic agent is allowed to buy (shortsell) at most an amount of π^i_{upper} (π^i_{lower}) of stock i. Suppose first that u is linear. We then have that

$$f(s, z, \tilde{z}) = \inf_{\pi_{\text{lower}} \le \pi \le \pi_{\text{upper}}} \Bigg\{ -\pi b_s + g(s, z - \pi\sigma_s, \tilde{z} - \pi\beta_s) \Bigg\}. \tag{5.1}$$

The function f consists of a penalty (the term with plus sign) minus a reward. Denote by π_s^* the optimal strategy attaining the infimum in (5.1). The Karush-Kuhn-Tucker conditions imply that attaining the infimum in (5.1) is equivalent to the existence of Lagrange multipliers $0 \le \mu_s^*, \zeta_s^*$ taking values in $\mathbb{R}^n$ such that

$$0 = \mu_s^* - \zeta_s^* - b_s - \sigma_s \partial_z g(s, z - \pi_s^* \sigma_s, \tilde{z} - \pi_s^* \beta_s) - \partial_{\tilde{z}} g(s, z - \pi_s^* \sigma_s, \tilde{z} - \pi_s^* \beta_s) \cdot \beta_s, \tag{5.2}$$

where the integral is understood componentwise and equality holds with respect to elements in the subgradients. Furthermore, μ_s^*, ζ_s^* satisfy the complimentary conditions, i.e.,

$$\mu_s^{*,i}(\pi_s^{*,i} - \pi^i_{\text{upper}}) = 0 \text{ and } \zeta_s^{*,i}(\pi_s^{*,i} - \pi^i_{\text{lower}}) = 0, \text{ for } i = 1, \ldots, n, \tag{5.3}$$

where π_{upper} (π_{lower}) denotes the vector consisting of the components π^i_{upper} (π^i_{lower}).

Note that (5.2)-(5.3) is a convex optimization problem. In particular, (5.2) yields that the excess return must satisfy

$$b_s = (\mu_s^* - \zeta_s^*) - \sigma_s \partial_z g(s, Z_s - \pi_s^* \sigma_s, \tilde{Z}_s - \pi_s^* \beta_s) - \partial_{\tilde{z}} g(s, Z_s - \pi_s^* \sigma_s, \tilde{Z}_s - \pi_s^* \beta_s) \cdot \beta_s. \tag{5.4}$$

Hence, under linear utility, the excess return can be decomposed into three parts: The first term on the right-hand side of (5.4) is due to the trading constraints, the second term is an ambiguity premium due to model uncertainty about the Brownian motion, and the third term is an ambiguity premium due to model uncertainty about the jumps. Note that the Lagrange multiplier μ_s^* and ζ_s^* represent the sensitivity of f, the difference between penalty and reward, with respect to the upper and lower hedging constraints. Furthermore, $\sigma_s \partial_z g(s, Z_s - \pi_s^* \sigma_s, \tilde{Z}_s - \pi_s^* \beta_s)$ may be seen as the sensitivity of the overall penalty with respect to the uncertainty of the Gaussian part. Finally, $\partial_{\tilde{z}} g(s, Z_s - \pi_s^* \sigma_s, \tilde{Z}_s - \pi_s^* \beta_s) \cdot \beta_s$ may be seen as the sensitivity of the overall penalty with respect to the uncertainty of the jump part. Hence, (5.4) yields that the excess return is the sum of the agent's different sensitivities with respect to the constraints and to the two sources of ambiguity.

Next, let us look at the case of an exponential utility function as considered in Section 4.2, so that the economic agent is not only ambiguity averse but also risk averse. Using a similar argument as above, it may be seen that we obtain the following decomposition of the excess return:

$$\begin{aligned} b_s =& (\mu_s^* - \zeta_s^*) - \frac{\sigma_s Z_s^\intercal - \sigma_s \sigma_s^\intercal \pi_s^*}{\gamma} - \sigma_s \partial_z g(s, Z_s - \pi_s^* \sigma_s, \tilde{Z}_s - \pi_s^* \beta_s) \\ & - \partial_{\tilde{z}} g(s, Z_s - \pi_s^* \sigma_s, \tilde{Z}_s - \pi_s^* \beta_s) \cdot \beta_s - \left(\exp\left\{ \frac{\tilde{Z}_s - \pi_s^* \beta_s}{\gamma} \right\} - 1 \right) \cdot \beta_s. \end{aligned} \tag{5.5}$$

Note that the first term on the right-hand side of (5.5) is again due to the trading constraints. The second term is a risk premium due to the risk arising from the Brownian motion, and the third term is an ambiguity premium due to model uncertainty about the Brownian motion. Furthermore, the fourth term is an ambiguity premium due to model uncertainty about the jumps, and the fifth term is a risk premium due to the risk arising from the jump part.

Decompositions may also be obtained for a power or a logarithmic utility function. Multiplying both sides in (5.4)-(5.5) by $\sigma_s^\intercal (\sigma_s \sigma_s^\intercal)^{-1}$, one obtains similar decompositions for the 'market price of uncertainty'.

6 Numerical Implementation

6.1 Some Analytics

Reconsider the setting of Section 5, with $d = 1$. The generalization to higher dimensions is straightforward. Suppose further for simplicity that the jump component is time-homogeneous and features only finite activity jumps with degenerate jump size. In this case, we can integrate with respect to a Poisson process dN_t, instead of with respect to $\tilde{N}(dt, dx)$. Write $d\tilde{N}_t := dN_t - adt$, where a is the intensity of N under P. Note that, for fixed ω and t, β_t and $\tilde{Z}_t$ now correspond to real numbers and not to functions. We furthermore assume that b, σ, and β do not depend on ω and t. Let us look at the case of exponential utility and consider the set $C = \{q \in \mathbb{R} \big| |q|_\infty \leq \lambda\}$. Furthermore, let $D = \{\psi | d^- \leq \psi \leq d^+\}$. The definition of D implies that for the new intensity a^Q, we have $(1 + d^-)a \leq a^Q = (1 + \psi)a \leq (1 + d^+)a$. Then the driver function corresponding to the optimal portfolio choice is given by (cf. (4.10))

$$\begin{aligned} f(z, \tilde{z}) = \inf_{\pi_{\text{lower}} \leq \pi \leq \pi_{\text{upper}}} \Bigg\{ & - \pi b + \frac{1}{2\gamma} |z - \pi\sigma|^2 + \lambda |z - \pi\sigma| \\ & + \left(\gamma \big(\exp\{-\frac{1}{\gamma}(\pi\beta - \tilde{z})\} - 1 + \frac{1}{\gamma}(\pi\beta - \tilde{z}) \big) + \bar{g}_2(\pi, \tilde{z}) \right) a \Bigg\}, \end{aligned} \tag{6.1}$$

with $\bar{g}_2(\pi, \tilde{z}) := \gamma \max\Big(d^+(-1+\exp\{-\frac{1}{\gamma}(\pi\beta - \tilde{z})\}), d^-(-1+\exp\{-\frac{1}{\gamma}(\pi\beta - \tilde{z})\})\Big)$. To solve minimization problem (6.1), we take the derivative with respect to π, set it equal to zero, and divide by σ^2. This yields

$$\begin{aligned} 0 \in & -\frac{b}{\sigma^2} + \frac{1}{\gamma}(\pi^* - \frac{z}{\sigma}) - \frac{\lambda}{\sigma} sign(\frac{z}{\sigma} - \pi^*) + \frac{\beta}{\sigma^2} a \\ & - \frac{\beta}{\sigma^2} a \Big(1 + d^+ I_{\{\tilde{z} > \pi\beta\}} + d^- I_{\{\tilde{z} < \pi\beta\}} + I_{\{\tilde{z} = \pi\beta\}}[d^-, d^+]\Big) \exp\{-\frac{\pi^*\beta - \tilde{z}}{\gamma}\} := h_{z,\tilde{z}}(\pi^*). \end{aligned}$$

One easily verifies the following proposition:

Proposition 6.1 *The driver function f in (6.1) is explicitly obtained by plugging $\pi^* := (\pi_{\text{lower}} \vee h^{-1}_{z,\tilde{z}}(0)) \wedge \pi_{\text{upper}}$ into the right-hand side of (6.1), i.e., π^* solves the optimization problem.*

We note that in this example $h^{-1}_{z,\tilde{z}}(0)$ can be computed, for instance, by using Newton's algorithm for every $(z, \tilde{z}) \in \mathbb{R}^2$.

6.2 Algorithm

In the simplified setting of this section, we can write (4.6) as

$$Y_t = F - \int_t^T f(s, Z_s, \tilde{Z}_s) ds + \int_t^T Z_s dW_s + \int_t^T \tilde{Z}_s d\tilde{N}_s, \quad t \in [0, T]. \tag{6.2}$$

Assume that $F = H(W_T, N_T)$ for a function $H : \mathbb{R}^2 \to \mathbb{R}$. The discrete-time BS$\Delta$E corresponding to (6.2) is given by

$$Y_{ih} = Y_{(i+1)h} - f(ih, Z_{ih}, \tilde{Z}_{ih})h + Z_{ih}\Delta W_{(i+1)h} + \tilde{Z}_{ih}\Delta\tilde{N}_{(i+1)h}. \tag{6.3}$$

Taking conditional expectations on both sides, we obtain

$$Y_{ih} = \mathrm{E}\left[Y_{(i+1)h} | \mathcal{F}_{ih}\right] - f(ih, Z_{ih}, \tilde{Z}_{ih})h. \tag{6.4}$$

We solve (6.3)-(6.4) by backward recursion, using a 'Longstaff-Schwartz type' of regression. For similar (yet slightly different) approaches in the case of a Brownian filtration, see Bender and Steiner [6] and the references therein.

Define an equi-spaced time grid $\{0, h, 2h, 3h, \ldots, T\}$ consisting of $L+1$ points with $T = Lh$. Simulate M paths of the Brownian motion W and the Poisson process N, generating values $w_{i,k}$ and $n_{i,k}$, where $w_{i,k}$ ($n_{i,k}$) is the value of the k-th path of the Brownian motion (Poisson process) at time ih. Denote by $\Delta n_{(i+1),k} = n_{(i+1),k} - n_{i,k}$ and define $\Delta w_{(i+1),k}$ similarly. We aim to compute the corresponding $y_{i,k}$, and we know that, at maturity, $y_{T,k} = H(w_{T,k}, n_{T,k})$. For this purpose, we first compute approximations to $\mathrm{E}\left[Y_{(i+1)h} | \mathcal{F}_{ih}\right]$, Z_{ih} and $\tilde{Z}_{ih}$, depending on the simulated paths. This proceeds in the following way. It follows from (6.3)-(6.4) that

$$Y_{(i+1)h} - \mathrm{E}\left[Y_{(i+1)h} | \mathcal{F}_{ih}\right] = -Z_{ih}\Delta W_{(i+1)h} - \tilde{Z}_{ih}\Delta\tilde{N}_{(i+1)h}.$$

But this entails that we can obtain $\mathrm{E}\left[Y_{(i+1)h} | \mathcal{F}_{ih}\right]$, $-Z_{ih}$ and $-\tilde{Z}_{ih}$ as the argmin of the minimization problem

$$\min_{a_{ih}, b_{ih}, c_{ih}} \mathrm{E}\left[\left(Y_{(i+1)h} - a_{ih} - b_{ih}\Delta W_{(i+1)h} - c_{ih}\Delta\tilde{N}_{(i+1)h}\right)^2 | \mathcal{F}_{ih}\right], \tag{6.5}$$

with the minimum attained in $a^*_{ih} := \mathrm{E}\left[Y_{(i+1)h}|\mathcal{F}_{ih}\right]$, $b^*_{ih} := -Z_{ih}$ and $c^*_{ih} := -\tilde{Z}_{ih}$.

Since all the quantities involved are $\mathcal{F}_{ih}$ measurable and the problem is Markov, there exist functions $A, B, C : \mathbb{R}^2 \to \mathbb{R}$ such that $A(W_{ih}, N_{ih}) = \mathrm{E}\left[Y_{(i+1)h}|\mathcal{F}_{ih}\right]$, $B(W_{ih}, N_{ih}) := -Z_{ih}$ and $C(W_{ih}, N_{ih}) := -\tilde{Z}_{ih}$. We fix $K \in \mathbb{N}$ and assume that there exists constants $a_{j_1,j_2}, b_{j'_1,j'_2}, c_{\bar{j}_1,\bar{j}_2} \in \mathbb{R}$ such that $\mathrm{E}\left[Y_{(i+1)h}|\mathcal{F}_{ih}\right] \approx \sum_{j_1=0,j_2=0}^{K} a_{j_1,j_2} W_{ih}^{j_1} N_{ih}^{j_2}$, $-Z_{ih} \approx \sum_{j'_1=0,j'_2=0}^{K} b_{j'_1,j'_2} W_{ih}^{j'_1} N_{ih}^{j'_2}$, and $-\tilde{Z}_{ih} \approx \sum_{\bar{j}_1=0,\bar{j}_2=0}^{K} c_{\bar{j}_1,\bar{j}_2} W_{ih}^{\bar{j}_1} N_{ih}^{\bar{j}_2}$. Then (6.5) suggests to calculate the desired approximations to $\mathrm{E}\left[Y_{(i+1)h}|\mathcal{F}_{ih}\right]$, Z_{ih} and $\tilde{Z}_{ih}$, given our simulated paths and $y_{i+1,k}$, using the following algorithm:

$$\min_{a_{j_1,j_2}, b_{j'_1,j'_2}, c_{\bar{j}_1,\bar{j}_2}} \sum_{k=0}^{M} \Big(y_{(i+1),k} - \sum_{j_1=0,j_2=0}^{K} a_{j_1,j_2} w_{i,k}^{j_1} n_{i,k}^{j_2} - \sum_{j'_1=0,j'_2=0}^{K} b_{j'_1,j'_2} \left[w_{i,k}^{j'_1} n_{i,k}^{j'_2} \Delta w_{i+1,k} \right] - \sum_{\bar{j}_1=0,\bar{j}_2=0}^{K} c_{\bar{j}_1,\bar{j}_2} \left[w_{i,k}^{\bar{j}_1} n_{i,k}^{\bar{j}_2} (I_{\{\Delta n_{(i+1),k}=0\}} - ah) \right] \Big)^2. \tag{6.6}$$

Note that this is a linear least squares regression in the $3(K+1)^2$ constants $a_{j_1,j_2}, b_{j'_1,j'_2}, c_{\bar{j}_1,\bar{j}_2}$. (Of course, other choices of basis functions and other types of regressions are also possible.)

Denote the constants that attain the minimum in (6.6) by a^*, b^*, c^* and set $\mathrm{E}\left[Y_{(i+1)h}|\mathcal{F}_{ih}\right] \approx \sum_{j_1=0,j_2=0}^{K} a^*_{j_1,j_2} w_{i,k}^{j_1} n_{i,k}^{j_2}$, $Z_{ih} \approx -\sum_{j'_1=0,j'_2=0}^{K} b^*_{j'_1,j'_2} w_{i,k}^{j'_1} n_{i,k}^{j'_2}$, and $\tilde{Z}_{ih} \approx -\sum_{\bar{j}_1=0,\bar{j}_2=0}^{K} c^*_{\bar{j}_1,\bar{j}_2} w_{i,k}^{\bar{j}_1} n_{i,k}^{\bar{j}_2}$. Finally, by (6.4), one can then calculate $y_{i,k}$ by

$$y_{i,k} = \sum_{j_1=0,j_2=0}^{K} a^*_{j_1,j_2} w_{i,k}^{j_1} n_{i,k}^{j_2} - f\Big(ih, -\sum_{j'_1=0,j'_2=0}^{K} b^*_{j'_1,j'_2} w_{i,k}^{j'_1} n_{i,k}^{j'_2}, -\sum_{\bar{j}_1=0,\bar{j}_2=0}^{K} c^*_{\bar{j}_1,\bar{j}_2} w_{i,k}^{\bar{j}_1} n_{i,k}^{\bar{j}_2} \Big) h. \tag{6.7}$$

In the particular case of a Markovian setting, such as that considered in this section, one may also employ a finite difference method based on the corresponding PIDE, adopting e.g., the method and results of Barles, Buckdahn and Pardoux [4]. Because in full generality our setting does not require a Markovian structure, we have primarily focussed attention on the Monte Carlo results for illustration purposes.

6.3 Numerical Results: Verification, Misspecification and Stability

We show numerical results for various special cases of Subsection 6.1. We consider a European put option with strike price 2 and time-to-maturity of 0.5 years and $S_0 = \exp(-bT)$. We take $b = 0.04$, $\sigma = 0.2$, $a = 1$, $\beta = 0.03$, $\pi_{\text{upper}} = 10$ and $\pi_{\text{lower}} = 0$. The number of Monte Carlo simulations is 10,000. Table 1 displays Y_0 as a function of γ. We consider subsequently the cases of (i) no ambiguity ($\lambda = d_+ = d_- = 0$), no hedge; (ii) no ambiguity ($\lambda = d_+ = d_- = 0$), with hedge; (iii) Brownian ambiguity only ($\lambda = 0.05$, $d_+ = d_- = 0$), with hedge; (iv) jump ambiguity only ($\lambda = 0$, $d_+ = 0.5$, $d_- = -0.25$), with hedge; (v) both Brownian ambiguity and jump ambiguity ($\lambda = 0.05$, $d_+ = 0.5$, $d_- = -0.25$), with hedge. In the limit, as γ tends to infinity, the risk averse γ-exponential utility maximizer becomes risk neutral: (vi) no ambiguity ($\lambda = d_+ = d_- = 0$), risk neutrality (asymptote $\gamma = \infty$), no hedge; (vii) no ambiguity ($\lambda = d_+ = d_- = 0$), risk neutrality (asymptote $\gamma = \infty$), with hedge. The table shows clearly (verifies) that risk aversion and ambiguity aversion decrease the evaluation, and that hedging opportunities increase the evaluation.

Table 1: Numerical Results

γ	1	4.5	14	32.5	63	108.5	γ	1	4.5	14	32.5	63	108.5
(i)	0.965	0.973	0.975	0.975	0.975	0.975	(vi)	0.976	0.976	0.976	0.976	0.976	0.976
(ii)	1.080	1.116	1.156	1.167	1.171	1.173	(vii)	1.173	1.173	1.173	1.173	1.173	1.173
(iii)	1.074	1.097	1.130	1.141	1.145	1.147	(viii)	1.055	1.077	1.110	1.121	1.125	1.127
(iv)	1.062	1.087	1.122	1.132	1.136	1.138	(ix)	0.964	0.972	0.974	0.974	0.974	0.975
(v)	1.059	1.073	1.097	1.108	1.113	1.114							

In the absence of ambiguity, as in Becherer [5], now suppose that the agent misspecifies the drift b (of which we know it cannot be estimated even consistently in finite time). In particular, suppose that the true b is 1% smaller (3%) than what the agent accounts for (4%). Then, upon comparing (viii) no ambiguity ($\lambda = d_+ = d_- = 0$), with hedge and misspecification; and (ii), we observe that Y_0 is (already) affected by about 3 to 5% (depending on γ), on a short time horizon. (Y_0 decreases because the hedge becomes less attractive when the drift decreases.) By contrast, the ambiguity averse agent takes the uncertainty regarding b already into account. His evaluation is mildly (or even hardly) affected by about -1 to 2%, as we see from Table 1, (iii)-(v). Thus, we conclude that no-ambiguity models are sensitive to the restrictive underlying assumption of a known distribution: purely risk-based models are prone to model uncertainty and taking model uncertainty into account provides a more robust evaluation.

As a verification of numerical stability, we have cross checked our results obtained by Monte Carlo least squares regression with results obtained by the specific simple random walk based approximation method of Lejay, Mordecki and Torres [58], which corresponds to a finite difference approximation based on the associated PIDE as in Barles, Buckdahn and Pardoux [4]. Table 1, (ix) shows the corresponding results for the case of no ambiguity ($\lambda = d_+ = d_- = 0$), no hedge. These results turn out to agree almost exactly with the Monte Carlo least squares regression results in (i), as desired. As a second verification of numerical stability, we have also increased the number of simulations to $2 \times 10,000$. We find that this occasionally leads to only slight variations in the third decimals of the results in Table 1. We conclude that the obtained results are numerically stable. A full-fledged convergence analysis is, however, beyond the scope of the current paper.

A Appendix

A.1 Preliminaries

Let $\mathcal{X}$ be a Banach space and denote by $\mathcal{X}^*$ the topological dual of $\mathcal{X}$. For a convex function $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$, we define its subgradient as $\partial f(x) = \{x^* \in \mathcal{X}^* | f(y) - f(x) \geq x^*(y) - x^*(x) \text{ for all } y \in \mathcal{X}\}$. If the function has several arguments, then the subdifferential should be taken with respect to the components in which the function is convex. For example, if $f(t, x)$ is convex in x, then we define $\partial f(t, x) = \{x^* \in \mathcal{X}^* | f(t, y) - f(t, x) \geq x^*(y - x) \text{ for all } y \in \mathcal{X}\}$. We say that a function $f(x)$ is subdifferentiable if its subgradient is nonempty for every x. For a convex function $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ not identical infinity, we denote by $f^*(x^*) = \sup_{x \in \mathcal{X}}\{x^*(x) - f(x)\}$ the dual conjugate of f mapping from $\mathcal{X}^*$ to $\mathbb{R} \cup \{\infty\}$. Again, if the function has several arguments, the dual conjugate should be taken with respect to the components in which the function is convex. The next result can be found in Zalinescu [86], Theorem 2.4.2(iii).

Proposition A.1 *Let $f : \mathcal{X} \to \mathbb{R} \cup \{\infty\}$ be a convex function with a nonempty domain. Then for every $x_0^* \in \mathcal{X}^*$ and $x_0 \in \mathcal{X}$, the following statements are equivalent:*

(i) $x_0 \in \partial f^*(x_0^*)$;

(ii) $x_0^* \in \partial f(x_0)$;

(iii) $f(x_0) = \max_{x^* \in \mathcal{X}^*}\{x^*(x_0) - f^*(x^*)\} = x_0^*(x_0) - f^*(x_0^*)$;

(iv) $f^*(x_0^*) = \max_{x \in \mathcal{X}}\{x_0^*(x) - f(x)\} = x_0^*(x_0) - f(x_0)$.

Let us recall some definitions:

Definition A.2 *A predictable process $H : [0,T]\times\Omega \to \mathbb{R}$ is called locally integrable if $\int_0^T |H_s|^2 ds < \infty$ a.s. A $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}\setminus\{0\})$-measurable function $\tilde{H} : [0,T]\times\Omega\times\mathbb{R}\setminus\{0\} \to \mathbb{R}$ is called locally integrable if $\int_0^T \int_{[-1,1]\setminus\{0\}} |\tilde{H}_s(x)|^2 n_p(dx)ds < \infty$ and $\int_0^T \int_{\mathbb{R}\setminus[-1,1]} |\tilde{H}_s(x)| n_p(dx)ds < \infty$ a.s.*

Definition A.3 *We call a martingale M a BMO(P) if there exists a constant $c > 0$ such that*

$$\mathrm{E}\left[\langle M\rangle_T - \langle M\rangle_\sigma |\mathcal{F}_\sigma\right] \leq c, \quad |\Delta M_\sigma|^2 \leq c \text{ for all stopping times } \sigma.$$

Furthermore, we call $Z : [0,T]\times\Omega \to \mathbb{R}$ a BMO(P) process if Z is predictable and there exists a constant $C > 0$ such that for every stopping time σ we have $\mathrm{E}\left[\int_\sigma^T |Z_s|^2 ds |\mathcal{F}_\sigma\right] \leq C$. We call $\tilde{Z} : [0,T]\times\Omega\times\mathbb{R}\setminus\{0\} \to \mathbb{R}$ a BMO(P) function if $\tilde{Z}$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R}\setminus\{0\})$-measurable, bounded, and there exists a constant $C > 0$ such that for every stopping time σ we have $\mathrm{E}\left[\int_\sigma^T \int_{\mathbb{R}\setminus\{0\}} |\tilde{Z}_s(x)|^2 n_p(dx)ds|\mathcal{F}_\sigma\right] \leq C$.

If Z and $\tilde{Z}$ are in $L^2(dP\times ds)$ and $L^2(dP\times n_p(dx)\times ds)$, respectively, then $M_t = \int_0^t Z_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x)\tilde{N}(ds,dx)$ is a square-integrable martingale. Furthermore, if Z is a BMO(P) processes and $\tilde{Z}$ is a BMO(P) function, then M is a BMO(P) martingale. We need the following result, also known as Kazamaki's [53] criterion.

Theorem A.4 *If M is a BMO(P) and there exists a $\bar{\delta} > 0$ such that $\Delta M > -1 + \bar{\delta}$ then the stochastic exponential of M, $\mathcal{E}(M_t)$, is a uniformly integrable martingale. Furthermore, $\mathcal{E}(M_T) > 0$.*

Remark A.5 If f is real-valued and convex (in some of its components), then many of the 'usual' rules of differentiation apply for its subgradient; see, specifically, Theorem 2.4.2, (vi)-(viii) in Zalinescu [86].

Define

$$(q\cdot W)_t := \int_0^t q_s dW_s \quad \text{and} \quad (\tilde{Z}\cdot\tilde{N}_p)_t := \int_0^t \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x)\tilde{N}_p(ds,dx).$$

Furthermore, we write $\Phi(x) := \exp\{x\} - x - 1 \geq 0$ and $\tilde{\Phi}(x) := \exp\{|x|\} - |x| - 1$ for $x \in \mathbb{R}$. Define also $\Psi(x) := \Phi^*(x) = (1+x)\log(1+x) - x \geq 0$ for $x \geq -1$ and infinity else. Set $\tilde{\Psi}(x) := \tilde{\Phi}^*(x) = (1+|x|)\log(1+|x|) - |x|$. The Fenchel dual inequality implies that for all $x, y \in \mathbb{R}$,

$$xy \leq \Psi(x) + \Phi(y), \quad \text{and} \quad xy \leq \tilde{\Psi}(x) + \tilde{\Phi}(y). \tag{A.1}$$

We will also need the notion of an Orlicz heart.

Definition A.6 *The Orlicz heart $M^{\tilde{\Phi}}(n_p)$ corresponding to $\tilde{\Phi}$ is the Banach space of all functions $\tilde{z} \in L^0(n_p)$ such that for every $a > 0$, $\int_{\mathbb{R}\setminus\{0\}} \tilde{\Phi}(a\tilde{z}(x))n_p(dx) < \infty$.*

Remark A.7 It holds that $L^2(n_p) \cap L^\infty(n_p) \subset M^{\tilde{\Phi}}(n_p) \subset L^2(n_p)$. Furthermore, $L^2(n_p) \cap L^\infty(n_p)$ is dense in $M^{\tilde{\Phi}}(n_p)$; see Rao and Ren [74].

Remark A.8 The dual of $M^{\tilde{\Phi}}(n_p)$ is $L^{\tilde{\Psi}}(n_p)$; see Rao and Ren [74]. In particular, a function $f : L^{\tilde{\Psi}}(n_p) \to M^{\tilde{\Phi}}(n_p)$ is weak* lower semi-continuous if for all x_n converging $\sigma(L^{\tilde{\Psi}}(n_p), M^{\tilde{\Phi}}(n_p))$-weakly to x we have that $f(x) \leq \liminf_n f(x_n)$.

A.2 Proofs

Lemma A.9 *Suppose that $Q = \lambda Q_1 + (1-\lambda)Q_2$ with $\lambda \in [0,1]$. Then it holds that $\frac{dQ}{dP} = \mathcal{E}\Big((q\cdot W)_T + (\psi \cdot \tilde{N}_p)_T\Big)$ with $(q,\psi) = \bar{\lambda}(q_1,\psi_1) + (1-\bar{\lambda})(q_2,\psi_2)$ and $\bar{\lambda} = \frac{\lambda D_{1,t}}{\lambda D_{1,t}+(1-\lambda)D_{2,t}}$.*

Proof. For $i = 1,2$ denote $D_{i,t} := \mathrm{E}\left[\frac{dQ_i}{dP}|\mathcal{F}_t\right]$. For a fixed $\lambda \in [0,1]$ denote $\bar{D} := \lambda D_1 + (1-\lambda)D_2$. We write

$$
\begin{aligned}
&d\bar{D}_t \\
&= \lambda D_{1,t}\big[q_{1,t}dW_t + \int_{\mathbb{R}\setminus\{0\}} \psi_{1,t}(x)\tilde{N}_p(dt,dx)\big] + (1-\lambda)D_{2,t}\big[q_{2,t}dW_t + \int_{\mathbb{R}\setminus\{0\}} \psi_{2,t}(x)\tilde{N}_p(dt,dx)\big] \\
&= \bar{D}_t\Big\{\frac{\lambda D_{1,t}}{\bar{D}_t}q_{1,t} + (1-\frac{\lambda D_{1,t}}{\bar{D}_t})q_{2,t}dW_t \\
&\quad + \int_{\mathbb{R}\setminus\{0\}}\Big(\frac{\lambda D_{1,t}}{\bar{D}_t}\psi_{1,t}(x) + (1-\frac{\lambda D_{1,t}}{\bar{D}_t})\psi_{2,t}(x)\Big)\tilde{N}_p(dt,dx)\Big\} \\
&= \bar{D}_t\Big\{(\bar{\lambda}q_{1,t} + (1-\bar{\lambda})q_{2,t})dW_t + \int_{\mathbb{R}\setminus\{0\}}\Big(\bar{\lambda}\psi_{1,t}(x) + (1-\bar{\lambda})\psi_{2,t}(x)\Big)\tilde{N}_p(dt,dx)\Big\},
\end{aligned}
\tag{A.2}
$$

with $\bar{\lambda} = \frac{\lambda D_{1,t}}{\bar{D}_t} = \frac{\lambda D_{1,t}}{\lambda D_{1,t}+(1-\lambda)D_{2,t}}$. Therefore,

$$
\frac{d(\lambda Q_1 + (1-\lambda)Q_2)}{dP} = \bar{D}_T = \mathcal{E}\Big\{((\bar{\lambda}q_1 + (1-\bar{\lambda})q_2)\cdot W)_T + ((\bar{\lambda}\psi_1 + (1-\bar{\lambda})\psi_2)\cdot \tilde{N}_p)_T\Big\}.
$$

□

Proof of Theorem 3.1. For the equivalence of (i) and (ii), see, for instance, Cheridito and Kupper [18]. This relationship is, in fact, true in full generality in every dynamic setting. The part (iii) ⇒ (ii) for a general measure n_p is straightforward to see. In the case that there are only finitely many markers, (iii) ⇐ (ii) follows from Theorem 4.2 and Remark 4.2 in Tang and Wei [84]. That (3.3) then also holds for a general measure n_p, for all $Q \in A := \cup_{n=1}^{\infty} A_n$ with $A_n = \{Q \ll P||q_s| \leq n, (-1+1/n)(1\wedge|x|) \leq |\psi_s(x)| \leq n(1\wedge|x|),\quad \text{for } ds\times n_p(dx) \text{ all } s,x\}$, follows from Theorem 4.1 in Tang and Wei [84].

So, let us show that for a general n_p we have that, if the domain of the penalty function c has a nonempty interior relative to $\mathcal{Q}$, then under time-consistency (or Bellman's principle) (3.3) must even hold for all $Q \in B := \{Q \ll P|\psi_s \in L^2(n_p),\ \psi_s(x) \geq -(1\wedge|x|)$ for $ds \times$

$n_p(dx)$-a.s. all $s, x\}$. From Theorem 4.1 in Tang and Wei [84] we find that there exists a convex and lower semi-continuous function $r(s, q, \psi)$ such that for every $Q \ll P$ we have

$$c_t(Q) \leq \mathrm{E}_Q \left[\int_t^T r(s, q_s, \psi_s) ds \right]. \tag{A.3}$$

Furthermore, we have already seen that equality in (A.3) holds for every $Q \in A$. Fix a $Q \in B$. Clearly, '$\geq$' holds if $Q \notin \mathrm{dom}(c)$. Thus, what is left to prove, is that '$\geq$' holds in (A.3) for $Q \in \mathrm{dom}(c) \cap B$. Assume next that $Q \in \mathrm{ri}(\mathrm{dom}(c)) \cap B$ with corresponding (q, ψ). Then clearly there exists a sequence $(q^n, \psi^n) \in \mathrm{dom}(c) \cap A$ with corresponding measures Q^n, such that (a): q^n converges to q $dP \times ds$-a.s.; (b): for $dP \times dt$-a.s. all t and ω, $\psi_t^n(\omega)$ converges to $\psi_t(\omega)$ in $L^2(n_p)$; and (c): Q^n converges to Q in the norm of the Banach space. This yields

$$\begin{aligned} c_t(Q) &= \lim_n c_t(Q^n) \\ &= \lim_n \mathrm{E}_Q \left[\int_t^T r(s, q_s^n, \psi_s^n) ds \right] \\ &\geq \mathrm{E}_Q \left[\int_t^T \liminf_n r(s, q_s^n, \psi_s^n) ds \right] \geq \mathrm{E}_Q \left[\int_t^T r(s, q_s, \psi_s) ds \right], \end{aligned} \tag{A.4}$$

where we used the assumption that c is continuous on its relative interior in the first equality. The first inequality holds by the lemma of Fatou since r must be non-negative as c is non-negative. The last inequality holds since r is lower semi-continuous.

Finally, let us consider the case that $Q \in \mathrm{dom}(c) \cap B \backslash \mathrm{ri}(\mathrm{dom}(c))$. Choose $Q_0 \in \mathrm{ri}(\mathrm{dom}(c)) \cap A$. Then for every $n \in \mathbb{N}$ we have that $\hat{Q}^n := \frac{1}{n} Q_0 + (1 - \frac{1}{n}) Q \in \mathrm{ri}(\mathrm{dom}(c)) \cap B$. Furthermore, by Lemma A.9 $(\hat{q}^n, \hat{\psi}^n)$ is a convex combination of (q_0, ψ_0) and (q, ψ), and $\hat{q}^n$ converges to q $dP \times ds$-a.s., and for $dP \times ds$-a.s. all t and ω $\hat{\psi}_t^n(\omega)$ converges to $\psi_t(\omega)$ in $L^2(n_p)$. From the previous step we already know that (A.3) holds with equality for every $\hat{Q}^n$. Now we can replace in (A.4) Q^n with $\hat{Q}^n$, and (q^n, ψ^n) with the corresponding $(\hat{q}^n, \hat{\psi}^n)$. To see that the first equality in (A.4) holds, consider the function $h(s) := c_t(sQ_0 + (1 - s)Q)$ mapping from $\mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$. Clearly, h is finite on $[0, 1]$ and, since h is also convex and lower semi-continuous by Proposition 2.1.6 in [86] h is continuous on the closure of its domain. Therefore the first equality in (A.4) still holds for Q^n replaced by $\hat{Q}^n$. By construction of our sequence, the other equalities and inequalities also hold. Therefore, we obtain '$\geq$' in (A.3) also for $Q \in (\mathrm{dom}(c) \cap B) \setminus \mathrm{ri}(\mathrm{dom}(c))$. This completes the proof of the theorem. □

Lemma A.10 *Equation (3.6) holds.*

Proof. By definition $U_t(F) = \min_{Q \ll P} \mathrm{E}_Q \left[u(F) + \int_t^T \bar{r}(s, q_s, \psi_s) ds \middle| \mathcal{F}_t \right]$, where $\bar{r}$ is defined by

$$\bar{r}(s, q, \psi) = \begin{cases} 0, & \text{if} \quad (q, \psi) \in \Big(\{(q_{i,s}, \psi_{j,s}) | i, j \in \{1, \dots, L\}\} \Big); \\ \infty, & \text{otherwise.} \end{cases}$$

We need to show that

$$\min_{Q \ll P} \mathrm{E}_Q \left[u(F) + \int_t^T \bar{r}(s, q_s, \psi_s) ds \middle| \mathcal{F}_t \right] = \min_{Q \ll P} \mathrm{E}_Q \left[u(F) + \int_t^T r(s, q_s, \psi_s) ds \middle| \mathcal{F}_t \right], \tag{A.5}$$

with r given by

$$r(s,q,\psi) = \begin{cases} 0, & \text{if} \quad (q,\psi) \in \operatorname{conv}\Big(\{(q_{i,s},\psi_{j,s}) | i,j \in \{1,\dots,L\}\}\Big); \\ \infty, & \text{otherwise.} \end{cases}$$

We may assume without loss of generality that $r(s,0,0) = 0$. Now clearly '$\geq$' in (A.5) holds, while '$\leq$' follows from Lemma A.30b) below (with $(Y,Z,\tilde{Z}) = (Y',Z',\tilde{Z}')$ and g as in Example 4.2(4)). □

To prove Theorem 4.1 we need the following inequalities:

Lemma A.11 *The following inequalities hold for all $C,\alpha,\lambda > 0$:*

$$\exp\left\{\frac{x}{\alpha}\right\} - \frac{x}{\alpha} - 1 \quad \leq \quad \frac{x^2}{\alpha^2}\exp\left\{\frac{C}{\alpha}\right\}, \qquad \textit{for all} \quad x \in [-C,C]; \tag{A.6}$$

$$x^2 \quad \leq \quad 2\alpha^2 e^{C/\alpha}\left[\exp\left\{\frac{x}{\alpha}\right\} - \frac{x}{\alpha} - 1\right], \qquad \textit{for all} \quad x \in [-C,C]; \tag{A.7}$$

$$|e^{x/\lambda} - 1| \quad \leq \quad e^{C/\lambda}\frac{|x|}{\lambda}, \qquad \textit{for all} \quad x \in (-\infty, C]. \tag{A.8}$$

Proof. As $e^{C/\alpha} > 1$, the first inequality can be seen from the sum expansion of the LHS (left-hand side) for $\frac{|x|}{\alpha} < 1$. For $\frac{|x|}{\alpha} \geq 1$, one easily verifies that already $\exp\{C/\alpha\}$ is an upper bound for the LHS in the first inequality. The second inequality for $x > 0$ can also be seen from the sum expansion of the RHS. For $x < 0$, one can compare the derivatives of the functions $f_1(x) = x^2$ and $f_2(x) = 2\alpha^2 e^{C/\alpha}[\exp\{\frac{x}{\alpha}\} - \frac{x}{\alpha} - 1]$. Then $f_1'(x) = 2x$ and $f_2'(x) = 2\alpha e^{C/\alpha}(\exp\{\frac{x}{\alpha}\} - 1)$. Now $f_1'(0) = 0 = f_2'(0)$ and $f_1'(-C) > f_2'(-C)$. As f_2' is convex and f_1' is linear this entails that $f_1'(x) \geq f_2'(x)$ for all $x \in [-C,0]$. Therefore, for $x \in [-C,0]$, $f_1(x) = -\int_x^0 f_1'(y)dy \leq -\int_x^0 f_2'(y)dy = f_2(x)$. This shows (A.7) for $x \in [-C,0]$. Finally, to see (A.8), define $\hat{f}_1(x) := |e^{x/\lambda} - 1|$ and $\hat{f}_2(x) := e^{C/\lambda}\frac{|x|}{\lambda}$. Then $\hat{f}_1(0) = 0 = \hat{f}_2(0)$. Furthermore, for $x \in (0,C]$, we have $\hat{f}_1'(x) = \frac{e^{x/\lambda}}{\lambda} \leq \frac{e^{C/\lambda}}{\lambda} = \hat{f}_2'(x)$. For $x < 0$, $\hat{f}_1'(x) = -\frac{e^{x/\lambda}}{\lambda} \geq -\frac{e^{C/\lambda}}{\lambda} = \hat{f}_2'(x)$. From these inequalities (A.8) follows. □

Corollary A.12 *Suppose that $\tilde{K}$ is in $L^{2,\infty}$ (see Section 2) and is bounded. Then*

$$\left\| \sup_t \int_{\mathbb{R}\setminus\{0\}} \Phi(\tilde{K}_t(x)) n_p(dx) \right\|_\infty < \infty,$$

$$\left\| \sup_t \int_{\mathbb{R}\setminus\{0\}} \Phi^2(\tilde{K}_t(x)) n_p(dx) \right\|_\infty < \infty.$$

Proof. The first statement follows immediately from (A.6) and the definition of $L^{2,\infty}$. Furthermore, if $\tilde{K}$ is bounded by a constant, say C, then (A.6) and the fact that $\Phi(x) \geq 0$ yields

$$\Phi^2(\tilde{K}_t(x)) \leq \tilde{K}_t^4(x)\exp\{2C\} \leq \tilde{K}_t^2(x) C^2 \exp\{2C\}.$$

As $\tilde{K} \in L^{2,\infty}$, the second statement follows. □

For a driver function $g(t,z,\tilde{z})$ of a BSDE with jumps, the following properties play an important role, while proving Theorem 4.1:

(a) $0 \le g$, and $g(t,0,0) = 0$ for all t.

(b) There exist $\bar{K}, K_2 > 0$ such that for all $t \in [0,T]$, all $z \in \mathbb{R}^{1\times d}$, and all $\tilde{z} \in M^{\tilde{\Phi}}(n_p)$ we have

$$g(t,z,\tilde{z}) \le \bar{K}(1+|z|^2) + K_2 \int_{\mathbb{R}\setminus\{0\}} \left(\exp\left\{ \frac{\tilde{z}(x)}{K_2} \right\} - \frac{\tilde{z}(x)}{K_2} - 1 \right) n_p(dx).$$

(c) (i) For every t, and $(z,\tilde{z}) \in \mathbb{R}^d \times (M^{\tilde{\Phi}}(n_p) \cap L^\infty(n_p))$ we have that $g(t,z,\tilde{z})$ is convex and subdifferentiable (in the space $\mathbb{R}^d \times M^{\tilde{\Phi}}(n_p)$). We also write $\partial g(t,z,\tilde{z}) = (\partial_z g(t,z,\tilde{z}), \partial_{\tilde{z}} g(t,z,\tilde{z}))$ to distinguish the different components of the subgradient

(ii) g has a modification such that for every $z, \tilde{z}$ the mapping $(t,\omega) \to g(t,\omega,z,\tilde{z})$ is predictable.

(d) For every $C > 0$, there exists a BMO(P) process, H, and a constant $\bar{K}_1 > 0$ such that for every t, z, ω, and all $\tilde{z} \in M^{\tilde{\Phi}}(n_p)$ bounded by C we have that

$$|q| \le H_t(\omega) + \bar{K}_1|z|, \text{ for all } q \in \partial_z g(t,\omega,z,\tilde{z}).$$

(e) For every $C > 0$, there exists a BMO(P) function $\tilde{H}$ and $\bar{K}_2, \epsilon > 0$ such that for every t, ω, z, and all $\tilde{z} \in M^{\tilde{\Phi}}(n_p)$ bounded by C, we have

$$|\psi| \le \tilde{H}_t(\omega) + \bar{K}_2|\tilde{z}|, \quad \text{ and } \quad (-1+\epsilon) \le \psi, \text{ for every } \psi \in \partial_{\tilde{z}} g(t,\omega,z,\tilde{z}).$$

We will see later that assumption (a) may be relaxed and assumption (b) may be replaced by

(b') For every $C > 0$, there exists $K'' > 0$ such that for all t, all z, and all $\tilde{z} \in M^{\tilde{\Phi}}(n_p)$ bounded by C we have

$$g(t,z,\tilde{z}) \quad \le \quad K'' \left(1 + |z|^2 + \int_{\mathbb{R}\setminus\{0\}} |\tilde{z}(x)|^2 n_p(dx) \right). \tag{A.9}$$

Remark A.13 Note that assumptions (b'), (d), and (e) are generalizations of Kobylanski's [55] quadratic growth conditions to a setting with infinite activity jumps.

Remark A.14 In case that n_p is finite, the boundedness of $\tilde{z}$ by a fixed constant also implies the boundedness of $\int_{\mathbb{R}\setminus\{0\}} |\tilde{z}(x)|^2 n_p(dx)$. Therefore, in this case in order for (b') to hold it is sufficient that for all $\tilde{z}$ bounded by C we have that $g(t,z,\tilde{z}) \le K''(1+|z|^2)$, where K'' may depend on C.

Remark A.15 Conditions (b')-(c) may be assumed to always hold in the case that n_p is finite and g is of the form

$$g(t,z,\tilde{z}) := g_1(t,z) + \int_{\mathbb{R}\setminus\{0\}} G_2(t,\tilde{z}(x)) n_p(dx), \tag{A.10}$$

for $g_1 : [0,T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}_0^+$ being convex in z and satisfying quadratic growth conditions and a jointly continuous, convex function $G_2 : [0,T] \times \mathbb{R} \to \mathbb{R}_0^+$. This may be seen as $\tilde{Z}$ arising in our

BSDEs will always be bounded by a constant depending on the terminal conditions. Therefore, G_2 may be modified outside a compact set which can be fixed throughout the paper, implying that condition (b) holds. Clearly, g is then also finite valued on $\mathbb{R}^d \times L^\infty(n_p)$ and condition (c) holds.

The case of a finite n_p and g having the structure given in (A.10) was also considered in Becherer [5].

Lemma A.16 *Under the assumptions (H1)-(H3), g defined in (4.2) satisfies (a)-(e).*

Proof. (a): By (4.2), $g(t,0,0) = -\inf_{q\in\mathbb{R}^d,\psi\in L^{\tilde{\Psi}}(n_p)} r(t,q,\psi) = 0$, where the last equality is satisfied by (H1). As $0 \in \partial r(t,0,0)$, by Proposition A.1 we get that $0 \in \partial g(t,0,0)$. In particular, the convex function $g(t,\cdot,\cdot)$ for every t has its global minimum in $(0,0)$. It follows that $g \geq 0$.

(b): We write

$$
\begin{aligned}
g(t,z,\tilde{z}) &= \sup_{q\in\mathbb{R}^d,\psi\in L^{\tilde{\Psi}}(n_p)} \{zq + \tilde{z}\cdot\tilde{\psi} - r(t,q,\psi)\} \\
&\leq K_1 + \sup_{q\in\mathbb{R}^d,\psi\in L^{\tilde{\Psi}}(n_p)} \left\{ zq + \tilde{z}\cdot\tilde{\psi} - K_2\left(|q|^2 + \int_{\mathbb{R}\setminus\{0\}} \Psi(\psi(x))n_p(dx)\right)\right\} \\
&\leq K_1 + \frac{|z|^2}{2K_2} + \int_{\mathbb{R}\setminus\{0\}} \sup_{y\in\mathbb{R}}[\tilde{z}(x)y - K_2\Psi(y)]n_p(dx) \\
&= K_1 + K_2\left[\frac{|z|^2}{2K_2^2} + \int_{\mathbb{R}\setminus\{0\}} \Phi\left(\frac{\tilde{z}(x)}{K_2}\right)n_p(dx)\right],
\end{aligned}
$$

where we used (H2) in the first inequality.

(c)(i): Clearly, $g(t,\cdot,\cdot)$ is convex. By (b), it also is a real-valued function. By Theorem 2.2.20 and Theorem 2.4.12 in Zalinescu [86] this implies that for every t, $g(t,\cdot,\cdot)$ is continuous and subdifferentiable even on the entire space $M^{\tilde{\Phi}}(n_p)$.

(c)(ii): As $g(t,\cdot,\cdot)$ is subdifferentiable this follows from a measurable selection theorem, see for instance Aumann [3].

(d): By Proposition A.1, we have for t,ω, z and $\tilde{z}$ that $(q,\psi) \in \partial g(t,\omega,z,\tilde{z})$ if and only if $(z,\tilde{z}) \in \partial r(t,\omega,q,\psi)$. Therefore, (H3) yields $|z| \geq -\hat{K}_1 + \hat{K}_2|q|$. Thus, indeed $|q| \leq \hat{K}_1 + \frac{|z|}{\hat{K}_2}$.

(e): For t,ω, and $\tilde{z}$ bounded by C, choose (q,ψ) with $(q,\psi) \in \partial g(t,\omega,z,\tilde{z})$. By Proposition A.1, we have then that $(z,\tilde{z}) \in \partial r(t,\omega,q,\psi)$. Therefore, (H3) yields

$$|\tilde{z}| \geq -\tilde{K} + \hat{K}_3|\log(1+\psi)|.$$

Now as $\tilde{K}$ is uniformly bounded by a constant, say $\bar{C}$, we must have that ψ is uniformly bounded, and bounded away uniformly from 0. It follows that

$$|\psi| \leq \left|\exp\left(\frac{|\tilde{z}|+\tilde{K}}{\hat{K}_3}\right) - 1\right| \leq e^{(C+\bar{C})/\hat{K}_3}\frac{|\tilde{z}|+\tilde{K}}{\hat{K}_3},$$

where we applied (A.8) in the second inequality. This shows (e). □

Remark A.17 Actually we have even proved that if (H1)-(H3) holds, $g(t,\cdot,\cdot)$ is continuous on $(\mathbb{R}^d \times M^{\tilde{\Phi}}(n_p), |\cdot| \times |\cdot|_{L^{\tilde{\Phi}}(n_p)})$.

Lemma A.18 *Suppose that we start with a function $g(t, z, \tilde{z})$ and denote by r the corresponding dual conjugate, i.e., for $t \in [0, T]$ we set*

$$r(t, q, \psi) : \ = \sup_{z \in \mathbb{R}^{1\times d}, \tilde{z} \in M^{\tilde{\Phi}}(n_p)} \Big\{ zq + \tilde{z} \cdot \psi - g(t, z, \tilde{z}) \Big\}, \ \text{for } q \in \mathbb{R}^d, \quad \psi \in L^{\tilde{\Psi}}(n_p).$$

Then property (b) implies that there exist constants $K_1, K_2, K_3 > 0$ such that for all t, all q, and all $\psi \in L^{\tilde{\Psi}}(n_p)$ we have

$$r(t, q, \psi) \geq -K_1 + K_2 |q|^2 + K_3 \int_{\mathbb{R}\setminus\{0\}} \Psi(\psi(x)) n_p(dx).$$

Proof. The lemma can be proved similarly as Lemma A.16, (b). □

From the definition of variational preferences in (2.3), with $u = \text{id}$, and (H1) we get

$$U_t(F) = \operatorname{ess\,inf} \left\{ \mathrm{E}_Q \left[F + \int_t^T r(s, q_s, \psi_s) ds \Big| \mathcal{F}_t \right] \Big| Q \ll P \right\}, \tag{A.11}$$

which is the object under consideration in Theorem 4.1. For a measure $Q \ll P$, let $D_t = \mathrm{E}\left[\frac{dQ}{dP} | \mathcal{F}_t\right]$ and $\tau = \inf\{t \in [0, T] | D_t = 0\} \wedge T$.

Lemma A.19 *For $T > t \geq \tau$ we have that $D_t = 0$. Furthermore, if $\tau^* = \inf\{t > 0 | D_{t-} = 0\} \wedge T$ then $\tau = \tau^*$.*

Proof. From the martingale stopping theorem for $T > t$,

$$\mathrm{E}\left[D_t I_{\{t\geq\tau\}} + D_t I_{\{t<\tau\}}\right] = \mathrm{E}\left[D_t\right] = 1 = \mathrm{E}\left[D_{t\wedge\tau}\right] = \mathrm{E}\left[D_t I_{\{t<\tau\}}\right].$$

Thus, $\mathrm{E}\left[D_t I_{\{t\geq\tau\}}\right] = 0$. As D is non-negative, the first part of the lemma follows. To see the second part note that the only possibility for $\tau \neq \tau^*$ is that, for fixed ω, the left-hand limit of the process D is zero at a time instance t, but D jumps (upwards) so that $D_{t-} = 0 < D_t$. In other words, for the increasing sequence of stopping times $\tau_m := \inf\{t > 0 | D_t \in (1/m, 0)\} \wedge (T - 1/m)$ we have that D jumps at $\tau' := \lim_m \tau_m$. However, as $\tau' > \tau_m$, τ' is a predictable stopping time, see Ch. III in Protter [73]. As the jumps of D are totally inaccessible, since they are induced by a (inhomogeneous) Poisson random measure, τ' a.s. cannot coincide with a jump time. □

The next two lemmas are the analogues of Lemma 2.1 and Proposition 2.1 in Delbaen, Hu and Bao [24]. They are proved there in a Brownian setting but the proofs also hold in our setting with obvious modifications:

Lemma A.20 *Suppose that (H1) holds. Then for any stopping time σ and $F \in L^\infty(\mathcal{F}_T)$,*

$$U_\sigma(F) = \operatorname{ess\,inf} \left\{ \mathrm{E}_Q \left[F + \int_\sigma^\tau r(s, q_s, \psi_s) ds \Big| \mathcal{F}_\sigma \right] \Big| Q \sim P \right\},$$

where $Q \sim P$ means that Q and P are equivalent in the sense that they share the same zero sets.

Lemma A.21 *Suppose that (H1) holds. Then for any $F \in L^\infty(\mathcal{F}_T)$ the process $U_t(F)$ defined by (A.11) has the following properties:*

(1) For all $Q \ll P$ we have that $U_t(F) + \int_0^{\tau\wedge t} r(s, q_s, \psi_s)ds$ is a Q-submartingale.

(2) If there is a probability measure $Q \ll P$ with

$$U_0(F) = \mathrm{E}_Q\left[F + \int_0^{\tau} r(s, q_s, \psi_s)ds\right],$$

then $U_t(F) + \int_0^{\tau\wedge t} r(s, q_s, \psi_s)ds$ is a Q-martingale.

As r is non-negative and $r(t, 0, 0) = 0$, clearly, for any $m \in \mathbb{R}$ and for any t, we have $U_t(m) = m$. Furthermore, U is monotone in the sense that $F \leq G$ implies that $U_t(F) \leq U_t(G)$. Therefore, for any $F \in L^\infty(\mathcal{F}_T)$ we have that $|U_t(F)| \leq ||F||_\infty$. Thus, we can apply the Doob-Meyer decomposition theorem to obtain that there exists a unique predictable increasing process A_t with $A_0 = 0$ and a local martingale M_t with $M_0 = 0$ such that

$$U_t(F) = U_0(F) + A_t - M_t. \tag{A.12}$$

For $k > 0$, set $C_k = \{Q \ll P | \mathrm{E}_Q\left[\int_0^T r(s, q_s, \psi_s)ds\right] \leq k\}$. Now (H2) entails that for every fixed k there is a constant $\bar{C} > 0$ such that for every $Q \in C_k$,

$$\mathrm{E}_Q\left[\int_0^T \left(|q_s|^2 ds + \int_{\mathbb{R}\setminus\{0\}} \Psi(\psi_s(x)) n_p(dx)\right) ds\right] \leq \bar{C}, \tag{A.13}$$

with $\Psi(x) = (1 + x)\log(1 + x) - x$ for $x \geq -1$. Denote $\tilde{N}_p^Q(dt, dx) := \tilde{N}_p(dt, dx) - (1 + \psi_t(x))n_p(dx)dt$. By Jacod and Shiryaev [49], Ch. 3, Th. 3.11 and Lemma 3.14 integrals of bounded locally integrable functionals with respect to $\tilde{N}_p^Q(dt, dx)$ give rise to local martingales with respect to Q. The next lemma shows that the local martingale in (A.12) is in fact a BMO(P) martingale. It prepares Theorem A.29, which is a key result for the proof of Theorem 4.1.

Lemma A.22 *Assume that the process J is a semi-martingale, bounded by a constant $\tilde{C}$, with Doob-Meyer decomposition $J = J_0 + A - M$ and A is increasing or decreasing. Then there exist a BMO(P) process Z and a BMO(P) function $\tilde{Z}$ such that*

$$M_t = \int_0^t Z_s dW_s + \int_0^t \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x) \tilde{N}_p(ds, dx). \tag{A.14}$$

Furthermore, for every $k \in \mathbb{N}$ and $Q \in C_k$, we have that $Z \in L^2(dQ \times ds)$ and $\tilde{Z} \in L^2(dQ \times n_p^Q(s, dx) \times ds)$.

Proof. We only prove the lemma for A increasing. The case that A is decreasing follows by considering $-J$. By the (local) martingale representation theorem, there exist predictable processes Z and $\tilde{Z}$ such that $M_t = (Z \cdot W)_t + (\tilde{Z} \cdot \tilde{N}_p)_t$. Note that the jumps of M are all totally inaccessible. Since A is predictable, a jump of M cannot coincide with a jump of A. As the jumps of J are uniformly bounded (since J is uniformly bounded), the jumps of M must be uniformly bounded too. In particular, M is locally square-integrable. This implies that we may choose Z and $\tilde{Z}$ in (A.14) such that

$$P\left[\int_0^T |Z_s|^2 ds < \infty\right] = 1 \quad \text{and} \quad P\left[\int_0^T \int_{\mathbb{R}\setminus\{0\}} |\tilde{Z}_s(x)|^2 n_p(dx) ds < \infty\right] = 1.$$

Since M cannot have a jump greater than $2||J||_{S^\infty}$ we also have that $|\tilde{Z}| \leq 2\tilde{C}$.

We first start our analysis for general $Q \in C_k$ with corresponding q and ψ. (Note that this includes the case that $Q = P$, since then $q = \psi = 0$.) For every $m \in \mathbb{N}$, define

$$\sigma_m = \inf \left\{ t \geq 0 | \int_0^t \int_{\mathbb{R}\setminus\{0\}} |Z_s|^2 ds \geq m \text{ and } \int_0^t \int_{\mathbb{R}\setminus\{0\}} |\tilde{Z}_s(x)|^2 n_p(dx) ds \geq m \right\}.$$

Then $\sigma_m \to \infty$ P-a.s. and therefore also Q-a.s. as m tends to infinity. Clearly, $(Z \cdot W^Q)_{t\wedge\sigma_m}$ is a square-integrable Q-martingale. That $(\tilde{Z} \cdot \tilde{N}^Q)_{t\wedge\sigma_m}$ is a square-integrable Q-martingale as well, may be seen from:

$$\begin{aligned}
&\mathrm{E}_Q\left[\int_0^{T\wedge\sigma_m} \int_{\mathbb{R}\setminus\{0\}} |\tilde{Z}_s(x)|^2 n_p^Q(s, dx) ds\right] \\
&\quad = \mathrm{E}_Q\left[\int_0^{T\wedge\sigma_m} \int_{\mathbb{R}\setminus\{0\}} |\tilde{Z}_s(x)|^2 (1 + \psi_s(x)) n_p(dx) ds\right] \\
&\quad \leq m + \mathrm{E}_Q\left[\int_0^{T\wedge\sigma_m} \int_{\mathbb{R}\setminus\{0\}} |\tilde{Z}_s(x)|^2 \psi_s(x) n_p(dx) ds\right] \\
&\quad \leq m + \mathrm{E}_Q\left[\int_0^{T\wedge\sigma_m} \int_{\mathbb{R}\setminus\{0\}} [\Phi(|\tilde{Z}_s(x)|^2) + \Psi(\psi_s(x))] n_p(dx) ds\right] \\
&\quad \leq m + \bar{C} + \mathrm{E}_Q\left[\int_0^{T\wedge\sigma_m} \int_{\mathbb{R}\setminus\{0\}} \Phi(|\tilde{Z}_s(x)|^2) ds\right] \\
&\quad \leq m + \bar{C} + \mathrm{E}_Q\left[\int_0^{T\wedge\sigma_m} \int_{\mathbb{R}\setminus\{0\}} \exp(4||J||^2_{S^\infty}) |\tilde{Z}_s(x)|^4 ds\right] \\
&\quad \leq m + \bar{C} + 4||J||^2_{S^\infty} \exp(4||J||^2_{S^\infty}) \mathrm{E}_Q\left[\int_0^{T\wedge\sigma_m} |\tilde{Z}_s(x)|^2 ds\right] \\
&\quad < m + \bar{C} + 4||J||^2_{S^\infty} \exp(4||J||^2_{S^\infty}) m,
\end{aligned}$$

with $\Phi(x) = e^x - x - 1$. In the first inequality, we used the definition of σ_m. In the second inequality, we applied (A.1). In the third inequality, we used (A.13). In the fourth inequality, we used (A.6) and that $|\tilde{Z}|^2$ is bounded by $4||J||^2_{S^\infty}$. In the fifth inequality, we applied again that $|\tilde{Z}|^2$ is bounded by $4||J||^2_{S^\infty}$. In the last inequality, we used the definition of σ_m. It follows that indeed $(\tilde{Z} \cdot \tilde{N}^Q)_{t\wedge\sigma_m}$ is a square-integrable Q-martingale. Therefore,

$$M_t^Q := M_t - \int_0^t Z_s q_s ds - \int_0^t \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x) \psi_s(x) n_p(dx) ds = (Z \cdot W^Q)_t + (\tilde{Z} \cdot \tilde{N}^Q)_t$$

is a locally square-integrable martingale with local stopping times σ_m. Next, choose C as in Lemma A.24 below (with $\bar{A} = 2||J||_{S^\infty}$). Note that, by (A.1),

$$-\psi(x)\tilde{z}(x) = -C\left(\psi(x)\frac{\tilde{z}(x)}{C}\right) \geq -C\Psi(\psi(x)) - C\Phi\left(\frac{\tilde{z}(x)}{C}\right). \tag{A.15}$$

Now, by Itô's generalized formula, for any stopping times σ and σ_m we have that Q-a.s.

$$
\begin{aligned}
\exp(J_{T\wedge\sigma_m}) = \exp(J_{\sigma\wedge\sigma_m}) - \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})dM_s + \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})dA_s \\
+ \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})\Big(\frac{1}{2}|Z_s|^2 + \int_{\mathbb{R}\setminus\{0\}} (\exp\{-\tilde{Z}_s(x)\} - 1 + \tilde{Z}_s(x))n_p(dx)\Big)ds \\
= \exp(J_{\sigma\wedge\sigma_m}) - \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})dM_s^Q + \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})dA_s \\
+ \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})\Big(\frac{1}{2}|Z_s|^2 - q_s Z_s - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x)\psi_s(x)n_p(dx) \\
+ \int_{\mathbb{R}\setminus\{0\}} (\exp\{-\tilde{Z}_s(x)\} - 1 + \tilde{Z}_s(x))n_p(dx)\Big)ds. \qquad \text{(A.16)}
\end{aligned}
$$

Taking conditional expectations on both sides in (A.16) yields

$$
\begin{aligned}
&\mathrm{E}_Q\left[\exp(J_{T\wedge\sigma_m})|\mathcal{F}_{\sigma\wedge\sigma_m}\right] \\
&= \exp(J_{\sigma\wedge\sigma_m}) + \mathrm{E}_Q\Big[\int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})dA_s + \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})\Big(\frac{1}{2}|Z_s|^2 - q_s Z_s \\
&\quad + \int_{\mathbb{R}\setminus\{0\}} [-\tilde{Z}_s(x)\psi_s(x) + \Phi(-\tilde{Z}_s(x))]n_p(dx)\Big)ds\Big|\mathcal{F}_{\sigma\wedge\sigma_m}\Big]. \qquad \text{(A.17)}
\end{aligned}
$$

As J is bounded by $\tilde{C}$, (A.17) entails that

$$
\begin{aligned}
&\mathrm{E}_Q\left[\exp(J_{T\wedge\sigma_m})|\mathcal{F}_{\sigma\wedge\sigma_m}\right] \\
&\geq \exp(-\tilde{C}) + \mathrm{E}_Q\Big[e^{-\tilde{C}}\Big(A_{T\wedge\sigma_m} - A_{\sigma\wedge\sigma_m}\Big) \\
&\quad + \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \exp(J_{s-})\Big(\frac{1}{4}|Z_s|^2 - 4|q_s|^2 + \int_{\mathbb{R}\setminus\{0\}} [-\tilde{Z}_s(x)\psi_s(x) + \Phi(-\tilde{Z}_s(x))]n_p(dx)\Big)ds\Big|\mathcal{F}_{\sigma\wedge\sigma_m}\Big] \\
&\geq \exp(-\tilde{C}) + \mathrm{E}_Q\Big[e^{-\tilde{C}}\Big(A_{T\wedge\sigma_m} - A_{\sigma\wedge\sigma_m}\Big) + \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \Big(\frac{e^{-\tilde{C}}}{4}|Z_s|^2 - 4e^{\tilde{C}}|q_s|^2 \\
&\quad + e^{J_{s-}} \int_{\mathbb{R}\setminus\{0\}} \Big\{ - C\Psi(\psi_s(x)) - C\Phi(\tilde{Z}_s(x)/C) + \Phi(-\tilde{Z}_s(x))\Big\}n_p(dx)\Big)ds\Big|\mathcal{F}_{\sigma\wedge\sigma_m}\Big] \\
&\geq \exp(-\tilde{C}) + \mathrm{E}_Q\Big[e^{-\tilde{C}}\Big(A_{T\wedge\sigma_m} - A_{\sigma\wedge\sigma_m}\Big) + \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m} \Big(\frac{e^{-\tilde{C}}}{4}|Z_s|^2 - 4e^{\tilde{C}}|q_s|^2 \\
&\quad + \int_{\mathbb{R}\setminus\{0\}} \big[- Ce^{\tilde{C}}\Psi(\psi_s(x)) + Be^{-\tilde{C}}|\tilde{Z}_s(x)|^2\big]n_p(dx)\Big)ds\Big|\mathcal{F}_{\sigma\wedge\sigma_m}\Big], \qquad \text{(A.18)}
\end{aligned}
$$

where $B > 0$ in the last inequality stems from Lemma A.24. In the first inequality, we used $|ab| \leq 4a^2 + \frac{b^2}{4}$ for the term $q_s Z_s$. The second inequality holds by (A.15). In the last inequality, we applied Lemma A.24 and the fact that $\tilde{Z}$ is bounded by $2||J||_\infty$. Now to see that Z is a BMO(P) process, note that for $Q = P$, we have that $q = \psi = 0$. (A.18) implies then that there exist a constant $C' > 0$ only depending on $\tilde{C}$ and B, such that for every stopping time σ and

σ_m,

$$\mathrm{E}\left[A_{T\wedge\sigma_m} - A_{\sigma\wedge\sigma_m} + \int_{\sigma\wedge\sigma_m}^{T\wedge\sigma_m}\Big(|Z_s|^2 + \int_{\mathbb{R}\setminus\{0\}}|\tilde{Z}_s(x)|^2 n_p(dx)\Big)ds\Big|\mathcal{F}_{\sigma\wedge\sigma_m}\right] \leq C'. \tag{A.19}$$

Choosing $\sigma = 0$ and letting m converge to infinity yields

$$\mathrm{E}\left[A_T + \int_0^T\Big(|Z_s|^2 + \int_{\mathbb{R}\setminus\{0\}}|\tilde{Z}_s(x)|^2 n_p(dx)\Big)ds\right] \leq C', \tag{A.20}$$

where we used the monotone convergence theorem. (Recall that $A_0 = 0$.) Now (A.20) implies that $Z \in L^2(dP \times ds)$ and $\tilde{Z} \in L^2(dP \times n_p(dx) \times ds)$. Therefore, M is a true martingale and we may choose $\sigma_m = T$. But then (A.19) yields that

$$\mathrm{E}\left[A_T - A_\sigma + \int_\sigma^T\Big(|Z_s|^2 + \int_{\mathbb{R}\setminus\{0\}}|\tilde{Z}_s(x)|^2 n_p(dx)\Big)ds\Big|\mathcal{F}_\sigma\right] \leq C'.$$

As $A_T - A_\sigma \geq 0$ (since A is increasing), it follows that Z is a BMO(P) process and $\tilde{Z}$ is a BMO(P) function.

For the second part of the lemma, let $Q \in C_k$. It follows from (A.13) and (A.18) with $\sigma = 0$ that there exists a constant $\hat{C}$ such that for every σ_m,

$$\mathrm{E}_Q\left[A_{T\wedge\sigma_m} + \int_0^{T\wedge\sigma_m}\Big(|Z_s|^2 + \int_{\mathbb{R}\setminus\{0\}}|\tilde{Z}_s(x)|^2 n_p(dx)\Big)ds\right] \leq \hat{C}. \tag{A.21}$$

Letting m converge to infinity and using the monotone convergence theorem (A.21) yields

$$\mathrm{E}_Q\left[A_T + \int_0^T\Big(|Z_s|^2 + \int_{\mathbb{R}\setminus\{0\}}|\tilde{Z}_s(x)|^2 n_p(dx)\Big)ds\right] \leq \hat{C}.$$

This shows that $A_T \in L^1(Q)$, $Z \in L^2(dQ \times ds)$ and $\tilde{Z} \in L^2(dQ \times n_p(dx) \times ds)$. What is left to show is that $\tilde{Z} \in L^2(dQ \times (1+\psi_s(x))n_p(dx) \times ds)$. First of all note that clearly $x^4 \leq 4||J||^2_{S^\infty}x^2$ for all $x \in \mathbb{R}$ with $|x| \leq 2||J||_{S^\infty}$. Hence, by (A.1) and (A.6),

$$\begin{aligned}|\tilde{Z}_s(x)|^2\psi_s(x) &\leq \Phi(|\tilde{Z}_s(x)|^2) + \Psi(\psi_s(x)) \\ &\leq \exp\{4||J||^2_{S^\infty}\}|\tilde{Z}_s(x)|^4 + \Psi(\psi_s(x)) \\ &= 4||J||^2_{S^\infty}\exp\{4||J||^2_{S^\infty}\}|\tilde{Z}_s(x)|^2 + \Psi(\psi_s(x)).\end{aligned} \tag{A.22}$$

Now we have already shown that $|\tilde{Z}_s(x)|^2 \in L^1(dQ \times n_p(dx) \times ds)$. On the other hand, $\Psi(\psi_s(x)) \in L^1(dQ \times n_p(dx) \times ds)$ because of (A.13). Therefore, it follows from (A.22) that $|\tilde{Z}_s(x)|^2(1+\psi_s(x)) \in L^1(dQ \times n_p(dx) \times ds)$, so that indeed $\tilde{Z} \in L^2(dQ \times n_p^Q(s,dx) \times ds)$. □

Remark A.23 Suppose that $J = J_0 + A - M$, where, rather than assuming that A is increasing, we assume that there exists a constant b such that $A_t + bt$ is increasing. This would be the case if A is given as the integral of a driver function bounded from below. In this case the conclusions of Lemma A.22 still hold. This can be seen by defining, $\bar{J}_t := J_0 + A_t + bt - M_t$. Then $A_t + bt$, the predictable part of $\bar{J}$, is increasing. As $\bar{J}$ is bounded, we can apply Lemma A.22 to $\bar{J}$ in order to obtain the integrability results on Z and $\tilde{Z}$.

Lemma A.24 *Let $\bar{A} > 0$. There exist $C, B > 0$ such that for all $x \in [-\bar{A}, \bar{A}]$,*

$$\Phi(-x) - C\Phi\left(\frac{x}{C}\right) \geq B|x|^2.$$

Proof. By (A.6)-(A.7),

$$\Phi(-x) - C\Phi\left(\frac{x}{C}\right) \geq \left(\frac{1}{2e^{\bar{A}}} - \frac{1}{C}\exp\left\{\frac{\bar{A}}{C}\right\}\right) x^2.$$

If we choose C large enough, then $B := \frac{1}{2e^{\bar{A}}} - \frac{1}{C}\exp\{\frac{\bar{A}}{C}\} > 0$. □

Proposition A.25 *For every $Q \ll P$ we have that*

$$H(Q|P) = \mathrm{E}_Q\left[\int_0^T \left(\frac{1}{2}|q_s|^2 + \int_{\mathbb{R}\setminus\{0\}} [(1+\psi_s(x))\log(1+\psi_s(x)) - \psi_s(x)]n_p(dx)\right)ds\right]. \quad \text{(A.23)}$$

Proof. As both sides are non-negative, it is sufficient to prove (A.23) if either the left- or the right-hand side is finite. First of all assume that we have Q such that $H(Q|P) < \infty$. If Q is equivalent to P, (A.23) corresponds to Proposition 9.10 in Cont and Tankov [19]. If Q is not equivalent to P, let $0 \leq \lambda \leq 1$ and define $Q^\lambda = \lambda Q + (1-\lambda)P$. It is not hard to see using the dominated convergence theorem that $H(Q^\lambda|P) \overset{\lambda\to 1}{\to} H(Q|P)$. On the other hand, a similar argument as in (3.2) yields that the density process of Q^λ is equal to $\mathcal{E}((q^\lambda \cdot W)_t + (\psi^\lambda \cdot \tilde{N}_p)_t)$ with

$$q_t^\lambda = \frac{\lambda D_t q_t}{\lambda D_t + (1-\lambda)} I_{\{t\leq\tau\}} \quad \text{and} \quad \psi_t^\lambda = \frac{\lambda D_t \psi_t}{\lambda D_t + (1-\lambda)} I_{\{t\leq\tau\}}.$$

Clearly, for every ω and s, $(q_s^\lambda(\omega))_\lambda$ and $(\psi_s^\lambda(\omega))_\lambda$ are increasing (decreasing) in $\lambda \in [0,1]$ on their respective positive (negative) parts. Furthermore, they converge to q_s and ψ_s, respectively, as λ tends to one. Therefore, indeed

$$\begin{aligned} H(Q|P) &= \lim_{\lambda\to 1} H(Q^\lambda|P) \\ &= \lim_{\lambda\to 1} \mathrm{E}_Q\left[\int_0^T \left(\frac{1}{2}|q_s^\lambda|^2 + \int_{\mathbb{R}\setminus\{0\}} \Psi(\psi_s^\lambda(x))n_p(dx)\right)ds\right] \\ &= \mathrm{E}_Q\left[\int_0^T \left(\frac{1}{2}|q_s|^2 + \int_{\mathbb{R}\setminus\{0\}} \Psi(\psi_s(x))n_p(dx)\right)ds\right], \end{aligned}$$

where we applied the monotone convergence theorem in the last equality.

Next, suppose that we have a Q with corresponding (q, ψ) such that the RHS in (A.23) is finite. Now clearly $\psi_{m,s}(x) = \psi_s(x) I_{\{\psi_s(x)\leq m\}} I_{\{|x|\geq 1/m\}}(x)$ is in $L^2(dQ \times n^Q(s,dx) \times ds)$.

Taking the logarithm of the Radon-Nikodym derivative yields that Q-a.s.

$$\begin{aligned}
\log\left(\frac{dQ}{dP}\right) &= \lim_m \log\left\{\mathcal{E}\Big(\int_0^T q_s dW_s + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \psi_{m,s}(x)\tilde{N}_p(ds,dx)\Big)\right\} \\
&= \lim_m \left\{\int_0^T q_s dW_s - \int_0^T \frac{1}{2}|q_s|^2 ds + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \psi_{m,s}(x)\tilde{N}_p(ds,dx)\right. \\
&\quad \left. + \int_0^T\int_{\mathbb{R}\setminus\{0\}} (\log(1+\psi_{m,s}(x)) - \psi_{m,s}(x))N_p(ds,dx)\right\} \\
&= \lim_m \left\{\int_0^T q_s dW_s^Q + \int_0^T \frac{1}{2}|q_s|^2 ds + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \psi_{m,s}(x)\tilde{N}_p^Q(ds,dx)\right. \\
&\quad + \int_0^T\int_{\mathbb{R}\setminus\{0\}} (\log(1+\psi_{m,s}(x)) - \psi_{m,s}(x))\tilde{N}_p^Q(ds,dx) \\
&\quad \left. + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \Big[\psi_{m,s}(x)\psi_s(x) + (1+\psi_s(x))\{\log(1+\psi_{m,s}(x)) - \psi_{m,s}(x)\}\Big] n_p(dx)ds\right\} \\
&= \lim_m \left\{\int_0^T q_s dW_s^Q + \int_0^T \frac{1}{2}|q_s|^2 ds + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_{m,s}(x))\tilde{N}_p^Q(ds,dx)\right. \\
&\quad \left. + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \Big[(1+\psi_s(x))\log(1+\psi_{m,s}(x)) - \psi_{m,s}(x)\Big] n_p(dx)ds\right\}, \qquad \text{(A.24)}
\end{aligned}$$

where we used in the second equality that, for fixed ω, by the definition of ψ_m, we have that $(1+\psi_s)\{\log(1+\psi_{m,s})-\psi_{m,s}\} \in L^1(n_p(dx)\times ds)$. Thus, $\log(1+\psi_{m,s})-\psi_{m,s} \in L^1(n_p^Q(s,dx)\times ds)$. In particular, $\int_0^T \int_{\mathbb{R}\setminus\{0\}}(\log(1+\psi_{m,s}(x)) - \psi_{m,s}(x))\tilde{N}_p^Q(ds,dx)$ is well-defined.

Now Lemma A.26 below yields that each of the processes

$$\begin{aligned}
M'_t : &= \int_0^t q_s dW_s^Q, \qquad M''_{m,t} := \int_0^t\int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_{m,s}(x))\tilde{N}_p^Q(ds,dx), \\
M''_t : &= \int_0^t\int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_s(x))\tilde{N}_p^Q(ds,dx), \qquad \text{(A.25)}
\end{aligned}$$

is a martingale, and M''_m converges in $L^1(Q)$ to M''. By switching to a subsequence, we may assume that the convergence holds a.s. Finally, by the monotone convergence theorem, the last term in (A.24) converges to $\int_0^T \int_{\mathbb{R}\setminus\{0\}} \Psi(\psi_s(x))n_p(dx)ds$ Q a.s. Thus, (A.24) yields that

$$\begin{aligned}
\log\left(\frac{dQ}{dP}\right) &= \int_0^T q_s dW_s^Q + \int_0^T \frac{1}{2}|q_s|^2 ds + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_s(x))\tilde{N}_p^Q(ds,dx) \\
&\quad + \int_0^T\int_{\mathbb{R}\setminus\{0\}} \Big[(1+\psi_s(x))\log(1+\psi_s(x)) - \psi_s(x)\Big] n_p(dx)ds. \qquad \text{(A.26)}
\end{aligned}$$

Taking the expectation in (A.26) with respect to Q, and using that by Lemma A.26 M' and M'' are martingales, (A.23) follows. □

Lemma A.26 *Let $Q \ll P$ be such that the RHS in (A.23) is finite. Then the stochastic processes M' and M'' defined in (A.25) are martingales. Furthermore, $M''_{m,T}$ converges in $L^1(Q)$ to M''_T.*

Proof. First of all note that, as the RHS in (A.23) is finite, we have $\mathrm{E}_Q\left[\int_0^T |q_s|^2 ds\right] < \infty$. Therefore, M' is a martingale. Let us prove that M'' is also a martingale. We write

$$\Psi(x) = (1+x)\log(1+x) - x \geq \frac{1}{6}(1+x)\log^2(1+x) \geq 0 \quad \text{for } -1 \leq x \leq e^2 - 1.$$

(This may be seen by noticing that at zero both sides are equal to zero and their derivatives are equal to zero, too. Furthermore, the second derivative of the LHS is larger than the second derivative of the RHS for $-1 \leq x \leq e^2 - 1$.) As the RHS in (A.23) is finite so that $\Psi(\psi_s(x)) \in L^1(dQ \times n_p(dx) \times dx)$, we obtain

$$(1+\psi_s(x))\log^2(1+\psi_s(x)) I_{\{\psi_s(x)\leq e^2-1\}}(x) \in L^1(dQ \times n_p(dx) \times ds).$$

This implies that

$$\log(1+\psi_s(x)) I_{\{\psi_s(x)\leq e^2-1\}}(x) \in L^2(dQ \times n_p^Q(s,dx) \times ds).$$

Furthermore, the positive (negative) parts of $\log(1+\psi_{m,s}(x)) I_{\{\psi_{m,s}(x)\leq e^2-1\}}(x)$ increase (decrease) to those of $\log(1+\psi_s(x)) I_{\{\psi_s(x)\leq e^2-1\}}(x)$ as m tends to infinity. By the monotone convergence theorem, this yields that

$$\log(1+\psi_{m,s}(x)) I_{\{\psi_{m,s}(x)\leq e^2-1\}}(x) \overset{m\to\infty}{\to} \log(1+\psi_s(x)) I_{\{\psi_s(x)\leq e^2-1\}}(x) \text{ in } L^2(dQ \times n_p^Q(s,dx) \times ds).$$

Therefore,

$$\begin{aligned}
\bar{M}_{m,t} :&= \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_{m,s}(x)) I_{\{\psi_{m,s}(x)\leq e^2-1\}}(x) \tilde{N}^Q(ds,dx) \\
\bar{M}_t :&= \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_s(x)) I_{\{\psi_s(x)\leq e^2-1\}}(x) \tilde{N}^Q(ds,dx)
\end{aligned}$$

are martingales in $L^2(Q)$, and $\bar{M}_{m,T}$ converges to $\bar{M}_T$ in $L^2(Q)$. Next, note that

$$(1+x)\log(1+x) \leq 2((1+x)\log(1+x) - x) \quad \text{for } x > e^2 - 1.$$

(This may be seen by noticing that the inequality holds for $x = e^2 - 1$, and that the derivative of the RHS is larger than the derivative of the LHS for $x > e^2 - 1$.) As the RHS in (A.23) is finite, it follows that $(1+\psi_s(x))\log(1+\psi_s(x)) I_{\{\psi_s(x)> e^2-1\}}(x) \in L^1(dQ \times n_p(dx) \times ds)$, so that

$$\log(1+\psi_s(x)) I_{\{\psi_s(x)> e^2-1\}}(x) \in L^1(dQ \times n_p^Q(s,dx) \times ds).$$

Moreover, $\log(1+\psi_{m,s}(x)) I_{\{\psi_{m,s}(x)> e^2-1\}}(x)$ increases to $\log(1+\psi_s(x)) I_{\{\psi_s(x)> e^2-1\}}(x)$ as m tends to infinity. By the monotone convergence theorem, it follows that

$$\log(1+\psi_{m,s}(x)) I_{\{\psi_{m,s}(x)> e^2-1\}}(x) \overset{m\to\infty}{\to} \log(1+\psi_s(x)) I_{\{\psi_s(x)> e^2-1\}}(x) \text{ in } L^1(dQ \times n_p^Q(s,dx) \times ds).$$

Consequently, by the definition of a compensator, the processes

$$\begin{aligned}
\hat{M}_{m,t} &:= \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_{m,s}(x)) I_{\{\psi_{m,s}(x)> e^2-1\}}(x) \tilde{N}^Q(ds,dx) \\
\hat{M}_t &:= \int_0^t \int_{\mathbb{R}\setminus\{0\}} \log(1+\psi_s(x)) I_{\{\psi_s(x)> e^2-1\}}(x) \tilde{N}^Q(ds,dx)
\end{aligned}$$

are both martingales, see, for instance, Jacod and Shiryaev [49], Ch. II, Th. 1.8(i). Furthermore, $\hat{M}_{m,T}$ converges in $L^1(Q)$ to $\hat{M}_T$. As $M'' = \bar{M} + \hat{M}$ and $M''_m = \bar{M}^1_m + \hat{M}_m$ the proposition now follows. □

Lemma A.27 *Let Z be a BMO(P) process and let $\tilde{Z}$ be a BMO(P) function. Suppose that g satisfies (c)-(e) and that we have a measure $Q \ll P$ with corresponding q and ψ such that $(q_t, \psi_t) \in \partial g(t, Z_t, \tilde{Z}_t)$, $dQ \times dt$ a.s. Then $Q \sim P$.*

Proof. Let $D_t = \mathrm{E}\left[\frac{dQ}{dP}|\mathcal{F}_t\right]$ and $\tau = \inf\{t \in [0,T]|D_t = 0\} \wedge T$. As $(q_t, \psi_t) \in \partial g(t, Z_t, \tilde{Z}_t)$ $dQ \times dt$ a.s., and since $\tilde{Z}$ is bounded, by properties (d)-(e), there exist constants $\bar{K}_1, \bar{K}_2 > 0$, a BMP(P) process H, and a BMO(P) function $\tilde{H}$, such that for Lebesgue-a.s. all $t \leq \tau$,

$$|q_t| \leq H_t + \bar{K}_1|Z_t| \quad \text{and} \quad |\psi_t| \leq \tilde{H}_t + \bar{K}_2|\tilde{Z}_t|. \tag{A.27}$$

Since, by assumption, Z is a BMO(P) process, and $\tilde{Z}$ and $\tilde{H}$ are BMO(P) functions, (A.27) entails that $(q_{t\wedge\tau})$ is a BMO(P) process and $(\psi_{t\wedge\tau})$ is a BMO(P) function. Furthermore, property (e) implies that $\psi_{t\wedge\tau} \geq -1 + \epsilon$ for an $\epsilon > 0$. But then $M_t := (q \cdot W)_{t\wedge\tau} + (\psi \cdot \tilde{N}_p)_{t\wedge\tau}$ is a BMO(P) martingale with $\Delta M_t \geq -1 + \epsilon$. Since $\frac{dQ}{dP} = \mathcal{E}(M_T)$, Theorem A.4 implies that $\frac{dQ}{dP} > 0$ and therefore also $\tau = T$, P-a.s. □

The next lemma can be proved in the same way using (A.27) and Theorem A.4 with τ replaced by T.

Lemma A.28 *Let Z and $\tilde{Z}$ be a BMO(P) process and a BMO(P) function, respectively. Suppose that g satisfies (c)-(e) and we have predictable q and ψ satisfying $(q_t, \psi_t) \in \partial g(t, Z_t, \tilde{Z}_t)$ $dP \times dt$ a.s. Then we have that the measure Q induced by $\frac{dQ}{dP} := \mathcal{E}((q \cdot W)_T + (\psi \cdot \tilde{N}_p)_T)$ is well-defined and equivalent to P.*

The following theorem generalizes Kobylanski's [55] existence and uniqueness result for quadratic drivers in a Brownian setting to an infinite activity jump setting.

Theorem A.29 *Suppose that g satisfies properties (a)-(e) above and let r be the dual conjugate of g. Define $U_t(F)$ by (A.11). Then $U_t(F)$ is the unique solution to the BSDE (4.3).*

Proof. From the decomposition (A.12) and Lemma A.22 it follows that, for every $Q \ll P$, there exist a BMO(P) process Z and a BMO(P) function $\tilde{Z}$ such that

$$\begin{aligned} dU_t(F) + r(t, q_t, \psi_t)dt &= dA_t - Z_t dW_t - \int_{\mathbb{R}\backslash\{0\}} \tilde{Z}_t(x)\tilde{N}_p(dt, dx) + r(t, q_t, \psi_t)dt \\ &= dA_t + \Bigg[- q_t Z_t - \int_{\mathbb{R}\backslash\{0\}} \tilde{Z}_t(x)\psi_t(x)n_p(dx) \\ &\qquad + r(t, q_t, \psi_t)\Bigg]dt - Z_t dW_t^Q - \int_{\mathbb{R}\backslash\{0\}} \tilde{Z}_t(x)\tilde{N}_p^Q(dt, dx), \end{aligned} \tag{A.28}$$

for Lebesgue-a.s. all $t \in [0, \tau]$. By Lemma A.21(1), $U_t(F) + \int_0^t r(s, q_s, \psi_s)ds$ is a Q-submartingale on $[0, T]$. (Recall that $\tau = T$ Q a.s.) Thus, for every Q,

$$dA_t \geq \left[q_t Z_t + \int_{\mathbb{R}\backslash\{0\}} \tilde{Z}_t(x)\psi_t(x)n_p(dx) - r(t, q_t, \psi_t)\right] dt, \quad Q\text{-a.s.} \tag{A.29}$$

By a measurable selection theorem we may choose predictable $(q_t, \psi_t) \in \partial g(t, Z_t, \tilde{Z}_t)$. By Lemma A.28, the corresponding measure Q is well-defined and $Q \sim P$. Plugging q and ψ into (A.29), we get $dA_t \geq g(t, Z_t, \tilde{Z}_t)dt$, Q-a.s. As $Q \sim P$ this implies that

$$dA_t \geq g(t, Z_t, \tilde{Z}_t)dt, \quad P\text{-a.s.} \tag{A.30}$$

Next, note that since g satisfies property (b), Lemma A.18 and Proposition A.25 yield that, for any $k > 0$, there exists a $k' > 0$ such that

$$C_k = \left\{ Q \ll P | \mathrm{E}_Q \left[\int_0^T r(s, q_s, \psi_s) ds \right] \le k \right\} \subset \left\{ Q \ll P | \mathrm{E} \left[\frac{dQ}{dP} \log \left(\frac{dQ}{dP} \right) \right] \le k' \right\}.$$

By the Dunford-Pettis theorem, this implies that C_k is weakly compact. Thus, for k large enough, the infimum in (A.11) is attained in a $Q \in C_k$ for $t = 0$. Again, let $D_t = \mathrm{E}\left[\frac{dQ}{dP} | \mathcal{F}_t\right]$ and $\tau = \inf\{t \ge 0 | D_t = 0\} \wedge T$. From (A.28) and the fact that, by Lemma A.21(2), $U_t(F) + \int_0^t r(s, q_s, \psi_s) ds$ is a Q-martingale on $[0, T]$ (as $\tau = T$ Q-a.s.), it follows that

$$dA_t = \left[q_t Z_t + \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_t(x) \psi_t(x) n_p(dx) - r(t, q_t, \psi_t) \right] dt, \quad Q\text{-a.s.} \tag{A.31}$$

By the definition of g, this implies $dA_t \le g(t, Z_t, \tilde{Z}_t) dt$, Q-a.s. Together with (A.30) we obtain that

$$dA_t = g(t, Z_t, \tilde{Z}_t) dt, \tag{A.32}$$

Q-a.s. By Proposition A.1, (A.31)-(A.32) entail that $(q_t, \psi_t) \in \partial g(t, Z_t, \tilde{Z}_t)$, $dQ \times dt$ a.s. (Notice that we need here the weak* lower semi-continuity of r so that duality holds for r and g.) By Lemma A.27, it follows that Q is an equivalent probability measure. Hence, the last equality holds P-a.s. Consequently, by (A.12), (A.14) and (A.32), $U_t(F)$ is indeed a solution to the BSDE (4.3). That $U_t(F)$ is the unique solution follows from Lemma A.30 below. This completes the proof. □

Denote $C_e = \left\{ Q \sim P | \mathrm{E}_Q \left[\int_0^T r(s, q_s, \psi_s) ds \right] < \infty \right\}$.

Lemma A.30 *Let $(Y', Z', \tilde{Z}')$ be a solution to a BSDE with driver function g satisfying properties (b)-(c) above such that $g(s, Z'_s, \tilde{Z}'_s)$ is uniformly bounded from below. (This is in particular the case if g is non-negative.) Then we have:*

(a) $Y'_t = U_t(F)$, where $U(F)$ is given by (A.11) with r being the dual conjugate of g.

(b) $U_t(F) = \min_{Q \in C_e} \mathrm{E}_Q \left[F + \int_t^T r(s, q_s, \psi_s) ds | \mathcal{F}_t \right]$, where the minimum is attained in $Q^* \in C_e$ with $(q^*_s, \psi^*_s) \in \partial g(s, Z'_s, \tilde{Z}'_s)$ $dP \times ds$ a.s.

Proof. Define $C = \left\{ Q \ll P | \mathrm{E}_Q \left[\int_0^T r(s, q_s, \psi_s) ds \right] < \infty \right\}$. Let $Q \in C$. We write

$$\begin{aligned}
Y'_t &= \mathrm{E}_Q \Bigg[F - \int_t^T g(s, Z'_s, \tilde{Z}'_s) ds + \int_t^T Z'_s dW_s + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}'_s(x) \tilde{N}_p(ds, dx) \mid \mathcal{F}_t \Bigg] \\
&= \mathrm{E}_Q \Bigg[F + \int_t^T \left[q_s Z'_s + \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}'_s(x) \psi_s(x) n_p(dx) - g(s, Z'_s, \tilde{Z}'_s) \right] ds \\
&\qquad + \int_t^T Z'_s dW^Q_s + \int_t^T \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}'_s(x) \tilde{N}^Q_p(ds, dx) \mid \mathcal{F}_t \Bigg] \\
&= \mathrm{E}_Q \Bigg[F + \int_t^T \left[q_s Z'_s + \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}'_s(x) \psi_s(x) n_p(dx) - g(s, Z'_s, \tilde{Z}'_s) \right] ds \mid \mathcal{F}_t \Bigg] \\
&\le \mathrm{E}_Q \Bigg[F + \int_t^T r(s, q_s, \psi_s) ds \mid \mathcal{F}_t \Bigg],
\end{aligned} \tag{A.33}$$

where we used in the first equality that Y'_t is $\mathcal{F}_t$-measurable. Note that the conditional expectation in the first equality is well-defined since Y' is bounded by the definition of a solution to a BSDE. The second and third equalities hold as $\int_0^t Z_s dW_s^Q$ and $\int_0^t \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}'_s(x)\tilde{N}_p^Q(ds,dx)$ are well-defined martingales. This may be seen since, by Lemma A.22 and Remark A.23 (with $J = Y'$ and $dA_t = g(t, Z_t, \tilde{Z}_t)dt$), we have that Z' and $\tilde{Z}'$ are in $L^2(dQ \times ds)$ and $L^2(dQ \times n_p^Q(s,dx) \times ds)$, respectively.

It follows from (A.33) and the fact that we can restrict the essential infimum in (A.11) to $Q \in C$, that

$$Y'_t \leq U_t(F).$$

Next, note that from Lemma A.22 and Remark A.23 it also follows that Z' is a BMO(P) process and $\tilde{Z}'$ is BMO(P) function. Next we choose predictable $(q_s^*, \psi_s^*) \in \partial g(s, Z'_s, \tilde{Z}'_s)$. q^* and ψ^* induce a stochastic exponential martingale $M_t := \mathcal{E}((q^* \cdot W)_t + (\psi^* \cdot \tilde{N}_p)_t)$. By Lemma A.28, $\frac{dQ^*}{dP} := M_T$ is an equivalent probability measure. Proceeding as in (A.33) with q^*, ψ^* and Q^* (where the inequality in (A.33) becomes an equality) yields

$$Y'_t = \mathrm{E}_{Q^*}\left[F - \int_t^T r(s, q_s^*, \psi_s^*)ds \mid \mathcal{F}_t\right].$$

Thus, by the definition of $U_t(F)$ in (A.11), we get $Y'_t \geq U_t(F)$. Therefore, indeed $Y' = U(F)$. As the essential infimum in (A.11) is always attained in a Q^* equivalent to P, part (b) also follows. □

Remark A.31 Theorem A.29 assumes that g satisfies (a)-(e). However, assumption (b) may be replaced by assumption (b'). This is seen as follows: As for a bounded terminal condition F, the corresponding $\tilde{Z}$ is bounded, it is sufficient that property (b) holds for $\tilde{z}$ bounded by an arbitrary fixed constant. (Of course, (a) and (c)-(e) must still hold.) The reason is that one may modify $g(t, z, \tilde{z})$ for $\tilde{z}$ with $L^\infty(n_p)$ norm greater than the specific bound, solve the BSDE with g_{modified}, and then observe that $g_{\text{modified}}(t, Z_t, \tilde{Z}_t)$ agrees with the original driver $g(t, Z_t, \tilde{Z}_t)$. In particular, it is sufficient that g satisfies property (b) for all $\tilde{z}$ bounded by a fixed constant. By Lemma A.11, this is equivalent to condition (b').

Proof of Theorem 4.1. Theorem 4.1 follows from Lemma A.16 and Theorem A.29. □

We now prepare the proof of Theorem 4.3.

Theorem A.32 *Suppose that g satisfies (a)-(e) or (a),(b'),(c)-(e). Let A_s and B_s be predictable and bounded processes, and let $\tilde{C}$ be a predictable and bounded functional in $L^{2,\infty}$ (see Section 2 for the definition). Then every BSDE with bounded terminal condition F and driver function*

$$\tilde{g}(t, z, \tilde{z}) := B_t + g(t, z - A_t, \tilde{z} - \tilde{C}_t) \tag{A.34}$$

has a unique solution $(Y, Z, \tilde{Z})$. Moreover, Z is a BMO(P) process, $\tilde{Z}$ is BMO(P) function, and we have

$$Y_t = \min_{Q \in C_e} \mathrm{E}_Q\left[F + \int_t^T \left[-B_s + A_s q_s + r(s, q_s, \psi_s) + \int_{\mathbb{R}\setminus\{0\}} \psi_s(x)\tilde{C}_s(x) n_p(dx)\right] ds \middle| \mathcal{F}_t\right]. \tag{A.35}$$

Proof. As property (b) implies property (b'), it is enough to prove the theorem in the case that (a),(b'),(c)-(e) hold. Define $\hat{g}(t,z,\tilde{z}) := g(t, z - A_t, \tilde{z} - \tilde{C}_t) - g(t, -A_t, -\tilde{C}_t)$. Now clearly, $\hat{g}$ satisfies property (c). That properties (b'),(d) and (e) hold is also not difficult to see. Unfortunately, $\hat{g}$ does not satisfy (a). Therefore, we have to define a new function, say g^{P^*}.

For this purpose, choose predictable $(q^*_s, \psi^*_s) \in \partial g(s, -A_s, -\tilde{C}_s)$. By property (d), q^* is bounded and by property (e) and the fact that $\tilde{C}$ is bounded and in $L^{2,\infty}$, we have that (i) $\psi^* \in L^{2,\infty}$, (ii) ψ^* is bounded by a constant, say $\bar{C}$, and (iii) $\psi^* \geq -1+\epsilon$. In particular, ψ^* is a BMO(P) function. Define a new reference measure P^* by setting $\frac{dP^*}{dP} = \mathcal{E}\Big((q^* \cdot W)_T + (\psi^* \cdot \tilde{N}_p)_T\Big)$. Next, define the driver function

$$g^{P^*}(t,z,\tilde{z}) := -zq^*_t - \int_{\mathbb{R}\setminus\{0\}} \frac{\psi^*_t(x)}{1+\psi^*(x)} \tilde{z}(x) n_p^{P^*}(dx) + \hat{g}(t,z,\tilde{z}).$$

By the definition of q^* and ψ^* we have that g^{P^*} has its minimum at $(z,\tilde{z}) = 0$. Furthermore, these minima are both equal to zero. Hence, g^{P^*} satisfies property (a). That g^{P^*} also satisfies (b'),(c)-(e) with respect to $n_p^{P^*}$ follows from the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, and the fact that q^* and ψ^* are bounded.

Therefore, by Theorem A.29 and Remark A.31, we may define $(\hat{Y}, Z, \tilde{Z})$ as the unique solution to the BSDE

$$\begin{aligned} d\hat{Y}_s &= g^{P^*}(s, Z_s, \tilde{Z}_s)ds - Z_s dW^{P^*}_s - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x) \tilde{N}_p^{P^*}(ds,dx) \\ \hat{Y}_T &= \hat{F}, \end{aligned} \tag{A.36}$$

with $\hat{F} := F - \int_0^T \big[B_s + g(s, -A_s, -\tilde{C}_s)ds$. (Note that $\hat{F}$ is bounded.) In particular,

$$\begin{aligned} d\hat{Y}_s &= \Bigg[-Z_s q^*_s + \hat{g}(s, Z_s, \tilde{Z}_s) - \int_{\mathbb{R}\setminus\{0\}} \frac{\psi^*_s(x)}{1+\psi^*_s(x)} \tilde{Z}_s(x) n_p^{P^*}(s,dx) \Bigg] ds \\ &\quad - Z_s dW^{P^*}_s - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x) \tilde{N}_p^{P^*}(ds,dx) \\ &= \hat{g}(s, Z_s, \tilde{Z}_s)ds - Z_s dW_s - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x) \tilde{N}_p(ds,dx). \end{aligned}$$

As $\hat{g}$ is uniformly bounded from below, Lemma A.22 and Remark A.23 yield that Z is a BMO(P) process and $\tilde{Z}$ is BMO(P) function. Therefore, under the measure P we have that Y is a solution to the BSDE with terminal condition $\hat{F}$ and driver function $\hat{g}(t,z,\tilde{z})$. The transformation

$$Y_t := \hat{Y}_t + \int_0^t \Big[B_s + g(s, -A_s, -\tilde{C}_t) \Big] ds \tag{A.37}$$

by the definition of $\tilde{g}$ (see (A.34)) and $\hat{g}$ yields the BSDE

$$\begin{aligned} dY_s &= \tilde{g}(s, Z_s, \tilde{Z}_s)ds - Z_s dW_s - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s(x) \tilde{N}_p(ds,dx), \\ Y_T &= F. \end{aligned} \tag{A.38}$$

Hence, the BSDEs (A.36) and (A.38) are equivalent. Now since (A.36) has a unique solution, (A.38) has a unique solution as well.

Finally, to see (A.35), note that the dual conjugates of $\hat{g}$ is given by

$$\hat{r}(s,q,\psi) = A_s q + r(s,q,\psi) + g(s,-A_s,-\tilde{C}_s) + \psi \cdot \tilde{C}_s.$$

As $\hat{g}$ is uniformly bounded from below, by Lemma A.30 (with terminal condition $\hat{F}$), we have

$$\hat{Y}_t = \min_{Q\in C_e} \mathrm{E}_Q\left[F - \int_0^T [B_s + g(s,-A_s,-\tilde{C}_s)]ds + \int_t^T \hat{r}(s,q_s,\psi_s)ds\Big|\mathcal{F}_t\right].$$

Together with (A.37), this yields (A.35). □

We also call (A.35) the dual representation of the solution to the corresponding BSDE.

Corollary A.33 *Suppose that $f(t,z,\tilde{z})$ is a driver function satisfying (b)-(e) or (b'),(c)-(e). Furthermore, suppose that f is bounded from below and that the argmin of f with respect to $z \in \mathbb{R}^d$ and $\tilde{z} \in L^2(n_p) \cap L^\infty(n_p)$, say A and $\tilde{C}$ (both depending on t and ω), are bounded. Suppose additionally that $\tilde{C} \in L^{2,\infty}$. Then for every bounded terminal condition there exists a unique solution, $(Y,Z,\tilde{Z})$ with Z being BMO(P) process and $\tilde{Z}$ being a BMO(P) function, for which the dual representation holds.*

Proposition A.34 *Suppose we have bounded terminal conditions F and G, and driver functions $\tilde{f}(t,z,\tilde{z})$ and $\tilde{g}(t,z,\tilde{z})$ being of the form (A.34), respectively. Then the solutions of the corresponding BSDEs, say Y^1 and Y^2, satisfy a comparison principle, i.e., if $F \le G$ and $\tilde{f} \ge \tilde{g}$ then $Y_1 \le Y_2$.*

Proof. Existence and uniqueness of Y_1 and Y_2 follow from Theorem A.32. The comparison principle follows directly from the dual representations. □

Lemma A.35 *We have that for every admissible π, $U_t(F + X_T^{(\pi)}) = U_t^{g^{(\pi)}}(F) + X_t^{(\pi)}$, where $U^{g^{(\pi)}}(F)$ is the unique solution to the BSDE with terminal condition F and driver function $g^{(\pi)}(t,z,\tilde{z})$ with*

$$g^{(\pi)}(t,z,\tilde{z}) := g(t, z - \pi_t\sigma_t, \tilde{z} - \pi_t\beta_t) - \pi_t b_t.$$

Proof. By Theorem A.32 (with $A = \pi\sigma$, $B = -\pi b$, and $\tilde{C} = \pi\beta$), the BSDE

$$\begin{aligned} d\hat{Y}_t^{(\pi)} &= g^{(\pi)}(s, Z_s^{(\pi)}, \tilde{Z}_s^{(\pi)})ds - Z_s^{(\pi)}dW_s - \int_{\mathbb{R}\setminus\{0\}} \tilde{Z}_s^{(\pi)}(x)\tilde{N}_p(ds,dx), \\ \hat{Y}_T^{(\pi)} &= F, \end{aligned}$$

has a unique solution, which we denote by $U^{g^{(\pi)}}(F)$ with BMO(P) process Z and BMO(P) function $\tilde{Z}$. Let $r^{(\pi)}$ be the dual conjugate of $g^{(\pi)}$ defined above. It is straightforward to verify that $r^{(\pi)}(s,q,\psi) = \pi_s b_s + \pi_s\sigma_s q + \int_{\mathbb{R}\setminus\{0\}} \psi(x)\pi_s\beta_s(x)n_p(dx) + r(s,q,\psi)$. (A.35) becomes then

$$U^{g^{(\pi)}}(F) = \min_{Q\in C_e} \mathrm{E}_Q\left[F + \int_t^T r^{(\pi)}(s,q_s,\psi_s)ds\Big|\mathcal{F}_t\right]. \tag{A.39}$$

As a result,

$$\begin{aligned}U_t(F+X_T^{(\pi)})-X_t^{(\pi)} &= \min_{Q\in C_e} \mathrm{E}_Q\Big[F+\int_t^T \pi_s\sigma_s dW_s^Q+\int_t^T\int_{\mathbb{R}\setminus\{0\}}\pi_s\beta_s(x)\tilde N_p^Q(ds,dx)\\&\quad+\int_t^T\Big[\pi_s b_s+\pi_s\sigma_s q_s+\int_{\mathbb{R}\setminus\{0\}}\psi_s(x)\pi_s\beta_s(x)n_p(dx)+r(s,q_s,\psi_s)\Big]ds\Big|\mathcal{F}_t\Big]\\&=\min_{Q\in C_e}\mathrm{E}_Q\left[F+\int_t^T r^{(\pi)}(s,q_s,\psi_s)ds\Big|\mathcal{F}_t\right]=U^{g^{(\pi)}}(F),\end{aligned}\tag{A.40}$$

where the first equality holds by (A.11) and the definition of $X^{(\pi)}$. The second equality holds because $\int_0^t \pi_s\sigma_s dW_s^Q$ is a Q-martingale as $Q\sim P$ and π and σ are uniformly bounded. To see that also $((\pi\beta)\cdot\tilde N_p^Q)_t$ is a Q-martingale notice that by our assumptions $\pi_s\beta_s$ is uniformly bounded by a constant, say C. Thus,

$$\begin{aligned}\mathrm{E}_Q\left[\int_0^T(\pi_s\beta_s(x))^2 n_p^Q(s,dx)ds\right] &= \mathrm{E}_Q\left[\int_0^T(1+\psi_s(x))(\pi_s\beta_s(x))^2 n_p(dx)ds\right]\\&\le K''+\mathrm{E}_Q\left[\int_0^T\psi_s(x)(\pi_s\beta_s(x))^2 n_p(dx)ds\right]\\&\le K''+\mathrm{E}_Q\left[\int_0^T[\Phi((\pi_s\beta_s(x))^2)+\Psi(\psi_s(x))]n_p(dx)ds\right]\\&\le K''+\mathrm{E}_Q\left[\int_0^T[\Phi(C|\pi_s\beta_s(x)|)+\Psi(\psi_s(x))]n_p(dx)ds\right]\\&\le K'''+\mathrm{E}_Q\left[\int_0^T\Psi(\psi_s(x))n_p(dx)ds\right]<\infty,\end{aligned}$$

where we used that the components of β are in $L^{2,\infty}$ in the first, and Corollary A.12 in the fourth inequality. The last term is smaller than infinity as $Q\in C_e$. It follows that $((\pi\beta)\cdot\tilde N_p^Q)_t$ is a Q-martingale. Now from (A.40) the lemma follows. □

Proof of Theorem 4.3. By Lemma A.35,

$$U_t(F+X_T^{(\pi)})=U_t^{g^{(\pi)}}(F)+X_t^{(\pi)},\tag{A.41}$$

where $U_t^{g^{(\pi)}}(F)$ is the unique solution to the BSDE with terminal condition F and driver function $g^{(\pi)}$. Consequently,

$$V_0(F)=\sup_{\pi\in\mathcal{A}}U_0(F+X_T^{(\pi)})=\sup_{\pi\in\mathcal{A}}\{U_0^{g^{(\pi)}}(F)+X_0^{(\pi)}\}=\sup_{\pi\in\mathcal{A}}U_0^{g^{(\pi)}}(F)+w_0.\tag{A.42}$$

By Corollary A.33 there exists a unique solution $(Y,Z,\tilde Z)$, satisfying a dual representation, to the BSDE with driver function f (defined in (4.5)) and terminal condition F. As Π is compact, we can now choose a predictable process π^* such that

$$f(s,Z_s,\tilde Z_s)=-\pi_s^*b_s+g(s,Z_s-\pi_s^*\sigma_s,\tilde Z_s-\pi_s^*\beta_s)=g^{(\pi^*)}(s,Z_s,\tilde Z_s).\tag{A.43}$$

If we could show that

$$\sup_{\pi\in\mathcal{A}}U_0^{g^{(\pi)}}(F)=Y_0,\tag{A.44}$$

then the theorem would follow from (A.42). Now '$\geq$' in (A.44) is seen since $Y_0 = U_0^{g^{(\pi^*)}}(F)$ by (A.43) and the definition of Y. On the other hand, '$\leq$' follows from Proposition A.34. Thus, we may infer that $Y_0 + w_0$ is the optimal value. Since $Y_0 + w_0 = U_0^{g^{(\pi^*)}}(F) + w_0 = U_0(F + X_T^{(\pi^*)})$, π^* is the optimal strategy. □

Proof of Theorem 4.5. Define

$$U_t^Q(F + X_T^{(\pi)}) = -\gamma \log \left(\mathrm{E}_Q \left[\exp \left\{ \frac{-F - X_T^{(\pi)}}{\gamma} \right\} \Big| \mathcal{F}_t \right] \right). \tag{A.45}$$

Note that U_t^Q may be seen as a mapping from $M^{\tilde{\Phi}}(P, \mathcal{F}_T)$ to $M^{\tilde{\Phi}}(P, \mathcal{F}_t)$ where $M^{\tilde{\Phi}}(P, \mathcal{F}_t)$ is the Orlicz space with the Young function $\tilde{\Phi}$ under the measure P and the filtration $\mathcal{F}_t$ for $t \in [0, T]$. As all exponential moments of $X_T^{(\pi)}$ exist it is well-known, see for instance Sec. 5.4, Cheridito and Li [16], that

$$U_t^Q(F + X_T^{(\pi)}) = \operatorname{ess\,inf}_{\bar{P} \ll P} \left\{ \mathrm{E}_{\bar{P}} \left[F + X_T^{(\pi)} | \mathcal{F}_t \right] + \gamma H_t(\bar{P}|Q) \right\}.$$

Furthermore, we may infer from Corollary 5.3 in Laeven and Stadje [56] that

$$U_t(F + X_T^{(\pi)}) := \operatorname{ess\,inf}_{Q \in M} U_t^Q(F) = \operatorname{ess\,inf}_{\bar{P} \ll P} \left\{ \mathrm{E}_{\bar{P}} \left[F + X_T^{(\pi)} | \mathcal{F}_t \right] + c_t(\bar{P}) \right\},$$

with $c_t(\bar{P}) = \gamma \operatorname{ess\,inf}_{Q \in M} H_t(\bar{P}|Q)$ being the weak* lower semi-continuous dual conjugate of U_t. For a $Q \in M$ with corresponding (q^Q, ψ^Q) denote

$$\frac{d\bar{P}}{dQ} = \mathcal{E}((\bar{q}^Q \cdot W^Q)_T + (\bar{\psi}^Q \cdot \tilde{N}_p^Q)_T), \quad \text{and} \quad \frac{d\bar{P}}{dP} = \mathcal{E}((\bar{q} \cdot W)_T + (\bar{\psi} \cdot \tilde{N}_p)_T).$$

It is not hard to see that $\bar{q} = \bar{q}^Q + q^Q$ and $1 + \bar{\psi} = (1 + \bar{\psi}^Q)(1 + \psi^Q)$. Hence, from Proposition A.25 we obtain

$$\begin{aligned}
c_t(\bar{P}) &= \operatorname{ess\,inf}_{Q \in M} \mathrm{E}_{\bar{P}} \left[\int_t^T \frac{1}{2} |\bar{q}_s^Q|^2 + \int_{\mathbb{R}\setminus\{0\}} [(1 + \bar{\psi}_s^Q(x)) \log(1 + \bar{\psi}_s^Q(x)) - \bar{\psi}_s^Q(x)] n_p^Q(dx) ds \Big| \mathcal{F}_t \right] \\
&= \mathrm{E}_{\bar{P}} \Bigg[\int_t^T \inf_{(q,\psi) \in C_s} \Bigg\{ \frac{1}{2} |\bar{q}_s - q|^2 \\
&\quad + \int_{\mathbb{R}\setminus\{0\}} \left[(1 + \bar{\psi}_s(x)) \log \Big(\frac{1 + \bar{\psi}_s(x)}{1 + \psi(x)} \Big) - (\bar{\psi}_s(x) - \psi(x)) \right] n_p(dx) \Bigg\} ds \Big| \mathcal{F}_t \Bigg] \\
&=: \mathrm{E}_{\bar{P}} \left[\int_t^T r(s, \bar{q}_s, \bar{\psi}_s) ds \Big| \mathcal{F}_t \right].
\end{aligned}$$

In the second equality note that '$\geq$' of course always holds while '$\leq$' holds, since, by our compactness assumptions, the inf on the RHS can be attained in a predictable (q_s, ψ_s) such that $(q, \psi) \in C$. Notice that our assumptions on C also imply that r, defined in the last equation, is weak* lower semi-continuous in $(\bar{q}, \bar{\psi})$ and satisfies (H1)-(H3).

The theorem would follow now directly from Theorem 4.3 if we could show that the dual of r with respect to (q, ψ), say r^*, is given by

$$g(s, z, \tilde{z}) := \frac{|z|^2}{2\gamma} + \bar{g}(s, z, \tilde{z}) + \gamma \int_{\mathbb{R}\setminus\{0\}} \left(\exp\left\{ \frac{\tilde{z}(x)}{\gamma} \right\} - 1 - \frac{\tilde{z}(x)}{\gamma} \right) n_p(dx),$$

with $\bar{g}$ defined in (4.8). To see this define

$$r^{q,\psi}(s,\bar{q},\bar{\psi}) := \frac{\gamma}{2}|\bar{q}-q|^2 + \gamma\int_{\mathbb{R}\setminus\{0\}}\left[(1+\bar{\psi}(x))\log\Big(\frac{1+\bar{\psi}(x)}{1+\psi(x)}\Big) - (\bar{\psi}(x)-\psi(x))\right]n_p(dx),$$

for $(q,\psi)\in C$. Note that by construction $r(s,\bar{q},\bar{\psi}) = \inf_{(q,\psi)\in C_s} r^{q,\psi}(s,\bar{q},\bar{\psi})$. We write

$$\begin{aligned} r^*(s,z,\tilde{z}) &= \sup_{(\bar{q},\bar{\psi})\in\mathbb{R}^d\times L^{\tilde{\Psi}}}\Big\{z\bar{q} + \tilde{z}\cdot\bar{\psi} - r(s,\bar{q},\bar{\psi})\Big\} \\ &= \sup_{(q,\psi)\in C_s}\ \sup_{(\bar{q},\bar{\psi})\in\mathbb{R}^d\times L^{\tilde{\Psi}}}\Big\{z\bar{q} + \tilde{z}\cdot\bar{\psi} - r^{q,\psi}(s,\bar{q},\bar{\psi})\Big\} \\ &= \sup_{(q,\psi)\in C_s}\left\{\frac{|z|^2}{2\gamma} + zq + \tilde{z}\cdot\psi + \gamma(1+\psi)\cdot\left(\exp\left\{\frac{\tilde{z}}{\gamma}\right\} - 1 - \frac{\tilde{z}}{\gamma}\right)\right\} = g(s,z,\tilde{z}), \end{aligned}$$

where the third equality can be seen through a simple direct computation. This finishes the proof of the theorem. □

Proof of Theorem 4.8. As the components of β are in $L^{2,\infty}$ and as, by assumption, $\rho\beta$ is bounded away from -1, clearly $\log(1+\rho\beta)\in L^{2,\infty}$ as well. Let

$$\begin{aligned} g^{(\rho)}(s,z,\tilde{z}) = g(s,z-\gamma\rho_s\sigma_s,\tilde{z}-\gamma\log(1+\rho_s\beta_s)) - \gamma\rho_s b_s + \frac{\gamma}{2}|\rho_s\sigma_s|^2 \\ - \int_{\mathbb{R}\setminus\{0\}}\gamma[\log(1+\rho_s\beta_s(x)) - \rho_s\beta_s(x)]n_p(dx). \end{aligned}$$

By Theorem A.32, there exists a unique solution, say $Y_t^{(\rho)}$, to the BSDE with terminal condition 0 and driver function $g^{(\rho)}$. It follows from (A.35) that

$$Y_t^{(\rho)} = \min_{Q\in C_e}\mathrm{E}_Q\left[\int_t^T r^{(\rho)}(s,q_s,\psi_s)ds\Big|\mathcal{F}_t\right], \tag{A.46}$$

with $r^{(\rho)}$ being the dual conjugate of $g^{(\rho)}$, given by

$$\begin{aligned} r^{(\rho)}(s,q,\psi):\ &=\ \gamma\Big(\rho_s b_s - \frac{1}{2}|\rho_s\sigma_s|^2 + \rho_s\sigma_s q + \frac{r(s,q,\psi)}{\gamma} \\ &\quad + \int_{\mathbb{R}\setminus\{0\}}\Big[\log(1+\rho_s\beta_s(x)) - \rho_s\beta_s(x) + \log(1+\rho_s\beta_s(x))\psi(x)\Big]n_p(dx)\Big). \end{aligned}$$

Using (4.12), we get (with U_t defined in (2.3))

$$\begin{aligned} &U_t(\gamma\log(X_T^{(\rho)})) - \gamma\log(X_t^{(\rho)}) \\ &= \min_{Q\in C_e}\mathrm{E}_Q\Big[\gamma\int_t^T\rho_s\sigma_s dW_s^Q + \gamma\int_t^T\int_{\mathbb{R}\setminus\{0\}}\log(1+\rho_s\beta_s(x))\tilde{N}_p^Q(ds,dx) + \int_t^T\Big[\gamma\rho_s b_s - \frac{\gamma}{2}|\rho_s\sigma_s|^2 \\ &\quad + \gamma\rho_s\sigma_s q_s + r(s,q_s,\psi_s) + \int_{\mathbb{R}\setminus\{0\}}\{\gamma\log(1+\rho_s\beta_s(x)) - \gamma\rho_s\beta_s(x) \\ &\quad + \gamma\log(1+\rho_s\beta_s(x))\psi_s(x)\}n_p(dx)\Big]ds\Big|\mathcal{F}_t\Big] \\ &= \min_{Q\in C_e}\mathrm{E}_Q\left[\int_t^T r^{(\rho)}(s,q_s,\psi_s)ds\Big|\mathcal{F}_t\right] = Y_t^{(\rho)}. \end{aligned}$$

The last equality holds by (A.46). Hence, $Y_t^{(\rho)} = U_t(\gamma \log(X_T^{(\rho)})) - \gamma \log(X_t^{(\rho)})$. This yields

$$V_0 = \sup_{\rho \in \mathcal{A}} U_0(\gamma \log(X_T^{(\rho)})) = \sup_{\rho \in \mathcal{A}} \{Y_0^{(\rho)} + \gamma \log(X_0^{(\rho)})\} = \sup_{\rho \in \mathcal{A}} Y_0^{(\rho)} + \gamma \log(w_0). \tag{A.47}$$

Now by Corollary A.33 there exists a unique solution, $(Y, Z, \tilde{Z})$, to the BSDE (4.15). Next we choose a predictable $\rho^* \in \tilde{C}$ such that

$$\begin{aligned} f(s, Z_s, \tilde{Z}_s) &= g(s, Z_s - \gamma \rho_s^* \sigma_s, \tilde{Z}_s - \gamma \log(1 + \rho_s^* \beta_s)) - \gamma \rho_s^* b_s + \frac{\gamma}{2} |\rho_s^* \sigma_s|^2 \\ &+ \int_{\mathbb{R} \setminus \{0\}} \big[- \gamma \log(1 + \rho_s^* \beta_s(x)) + \gamma \rho_s^* \beta_s(x) \big] n_p(dx) = g^{(\rho^*)}(s, Z_s, \tilde{Z}_s), \end{aligned}$$

where f was defined in (4.14). By the definition of Y in (4.15) this yields that $Y = Y^{(\rho^*)}$. If we could show that

$$\sup_{\rho \in \mathcal{A}} Y_0^{(\rho)} = Y_0, \tag{A.48}$$

then from (A.47) the theorem would follow. Now '$\geq$' in (A.48) follows as $Y_0 = Y_0^{(\rho^*)}$. On the other hand '$\leq$' follows from Proposition A.34. Thus, we may conclude that $V_0 = Y_0 + \gamma \log(w_0)$. Since $Y_0 + \gamma \log(w_0) = U_0(\gamma \log(X_T^{(\rho^*)}))$, ρ^* is the optimal strategy. □

Proof of Theorem 4.11. As the components of β are in $L^{2,\infty}$ and as, by assumption, $\rho\beta$ is bounded away from -1, a Taylor expansion of $(1 + \rho_s \beta_s)^\gamma$ and of $e^{\tilde{z}}$ yields that for every $\tilde{z} \in L^2(n_p) \cap L^\infty(n_p)$ we have

$$\Big[(1 + \rho_s \beta_s)^\gamma e^{\tilde{z}} - \tilde{z} - \rho_s \beta_s - 1 \Big] \in L^2(n_p) \cap L^\infty(n_p).$$

The main idea for the remainder of this proof is now to employ our existence results on BSDEs derived above and, on this basis, apply an approach similar in spirit to that of Hu, Imkeller and Müller [47], Müller [69] and Morlais [68]. Specifically, we will construct a family $(R^{\rho,(q,\psi)})_{\rho \in \mathcal{A}, Q^{(q,\psi)} \in \mathcal{Q}}$ with a trading strategy ρ^* and a probability measure Q^* (with corresponding (q^*, ψ^*)) such that

(i) $R_T^{\rho,(q,\psi)} = \frac{X_T^\gamma}{\gamma} \frac{dQ}{dP} = \frac{X_T^\gamma}{\gamma} \mathcal{E}\Big((q \cdot W)_T + (\psi \cdot \tilde{N}_p)_T \Big)$.

(ii) $R_0^{\rho,(q,\psi)} = R_0 = \frac{w_0^\gamma}{\gamma} \exp\{-Y_0\}$ does not depend on $(q, \psi) \in C$ and $\rho \in \mathcal{A}$.

(iii) $R_t^{\rho,(q^*,\psi^*)}$ is a P-supermartingale for all $\rho \in \mathcal{A}$.

(iv) $R_t^{\rho^*,(q,\psi)}$ is a P-submartingale for all $(q, \psi) \in C$.

(v) $R_t^{\rho^*,(q^*,\psi^*)}$ is a P-martingale.

We then get that

$$\mathrm{E}\left[R_T^{\rho,(q^*,\psi^*)} \right] \leq R_0 = \frac{w_0^\gamma}{\gamma} \exp\{-Y_0\} = \mathrm{E}\left[R_T^{\rho^*,(q^*,\psi^*)} \right] \leq \mathrm{E}\left[R_T^{\rho^*,(q,\psi)} \right], \text{ for all } \rho \in \mathcal{A}, (q, \psi) \in C.$$

Since this implies that R_0 is the optimal solution and ρ^* is the optimal strategy, we would have proved Theorem 4.11. To construct such a family $(R_t^{\rho,(q,\psi)})$ define

$$R_t^{\rho,(q,\psi)} := \frac{w_0^\gamma}{\gamma} \exp\Bigg\{ ((\gamma\rho\sigma + q)\cdot W)_t + ((\gamma\rho\beta+\psi)\cdot \tilde{N}_p)_t + \int_0^t \Big(\gamma\rho_s b_s - \frac{|q_s|^2 + \gamma|\rho_s\sigma_s|^2}{2}\Big) ds$$
$$- Y_t + \int_0^t \int_{\mathbb{R}\setminus\{0\}} [\gamma \log(1+\rho_s\beta_s(x)) - \gamma\rho_s\beta_s + \log(1+\psi_s(x)) - \psi_s(x)] N_p(ds,dx) \Bigg\},$$

where Y is the solution of the BSDE (4.18). Existence and uniqueness follow from Corollary A.33. Using Itô's generalized formula, for any strategy ρ and any $(q,\psi)\in C$ we have

$$\begin{aligned}
&R_t^{\rho,(q,\psi)} - R_0^{\rho,(q,\psi)}\\
&= \int_0^t R_{s-}^{\rho,(q,\psi)} (\gamma\rho_s\sigma_s + q_s + Z_s) dW_s + \int_0^t R_{s-}^{\rho,(q,\psi)} \int_{\mathbb{R}\setminus\{0\}} \big[\psi_s(x) + \tilde{Z}_s(x) + \gamma\rho_s\beta_s(x)\big] \tilde{N}_p(ds,dx)\\
&\quad + \int_{\mathbb{R}\setminus\{0\}} \Big\{\gamma\log(1+\rho_s\beta_s(x)) - \gamma\rho_s\beta_s(x) + \log(1+\psi_s(x)) - \psi_s(x)\Big\} N_p(ds,dx)\\
&\quad + \int_0^t R_{s-}^{\rho,(q,\psi)} \Big\{\gamma\rho_s b_s - \frac{1}{2}(\gamma|\rho_s\sigma_s|^2 + |q_s|^2) - f(s,Z_s,\tilde{Z}_s) + \frac{1}{2}|\gamma\rho_s\sigma_s + q_s + Z_s|^2\Big\} ds\\
&\quad + \int_0^t R_{s-}^{\rho,(q,\psi)} \int_{\mathbb{R}\setminus\{0\}} \Big[(1+\psi_s(x))(1+\rho_s\beta_s(x))^\gamma \exp(\tilde{Z}_s(x)) - 1 - \tilde{Z}_s(x)\\
&\qquad - \gamma\log(1+\rho_s\beta_s(x)) - \log(1+\psi_s(x))\Big] N_p(dx,ds)\\
&= \int_0^t R_{s-}^{\rho,(q,\psi)} (\gamma\rho_s\sigma_s + q_s + Z_s) dW_s\\
&\quad + \int_0^t R_{s-}^{\rho,(q,\psi)} \int_{\mathbb{R}\setminus\{0\}} \big[(1+\psi_s(x))(1+\rho_s\beta_s(x))^\gamma \exp(\tilde{Z}_s(x)) - 1\big] \tilde{N}_p(ds,dx)\\
&\quad + \int_0^t R_{s-}^{\rho,(q,\psi)} \Big\{\gamma\rho_s b_s - f(s,Z_s,\tilde{Z}_s) + \frac{\gamma(\gamma-1)}{2}|\rho_s\sigma_s|^2 + \frac{|Z_s|^2}{2} + \gamma\rho_s\sigma_s Z_s + (\gamma\rho_s\sigma_s + Z_s) q_s\Big\} ds\\
&\quad + \int_0^t R_{s-}^{\rho,(q,\psi)} \int_{\mathbb{R}\setminus\{0\}} \Big[(1+\psi_s(x))(1+\rho_s\beta_s(x))^\gamma \exp(\tilde{Z}_s(x)) - \tilde{Z}_s(x)\\
&\qquad - \gamma\rho_s\beta_s(x) - \psi_s(x) - 1\Big] n_p(dx)ds.
\end{aligned}$$

Therefore, $R^{\rho,(q,\psi)}$ satisfies: $dR^{\rho,(q,\psi)} = R_-^{\rho,(q,\psi)} dM^{\rho,(q,\psi)} + R_-^{\rho,(q,\psi)} dA^{\rho,(q,\psi)}$, with $A^{\rho,(q,\psi)}$ such that

$$dA_s^{\rho,(q,\psi)} := \Bigg(\gamma\rho_s b_s - f(s,Z_s,\tilde{Z}_s) + \gamma\rho_s\sigma_s Z_s + (\gamma\rho_s\sigma_s + Z_s) q_s + \psi_s \cdot ((1+\rho_s\beta_s)^\gamma \exp(\tilde{Z}_s) - 1)$$
$$+ \frac{\gamma(\gamma-1)}{2}|\rho_s\sigma_s|^2 + \frac{|Z_s|^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \Big[(1+\rho_s\beta_s(x))^\gamma \exp(\tilde{Z}_s(x)) - \tilde{Z}_s(x) - \gamma\rho_s\beta_s(x) - 1\Big] n_p(dx)\Bigg) ds,$$

and with $M_t^{\rho,(q,\psi)}$ denoting the local martingale $((\gamma\rho\sigma + q + Z)\cdot W)_t + \Big(\big[(1+\psi)(1+\rho\beta)^\gamma \exp(\tilde{Z}) - 1\big]\cdot \tilde{N}_p\Big)_t$. It follows that $R^{\rho,(q,\psi)}$ has the multiplicative form.

$$R_t^{\rho,(q,\psi)} = \mathcal{E}(M_t^{\rho,(q,\psi)}) \exp(A_t^{\rho,(q,\psi)}).$$

By Corollary A.33, Z is a BMO(P) process, and $\tilde{Z}$ is a BMO(P) function. In particular, by our assumptions on ψ and β, $\left[(1+\psi)(1+\rho\beta)^{\gamma}\exp(\tilde{Z})-1\right]$ is also a BMO(P) function bounded uniformly away from minus one. Furthermore, by our assumptions ρ, σ and q are uniformly bounded. Therefore, the Doléans-Dade exponential, $\mathcal{E}(M_t^{\rho,(q,\psi)})$, is a positive martingale. Next, choose a predictable saddle point $\rho^* \in \mathcal{A}$ and $(q^*, \psi^*) \in C$ such that

$$\begin{aligned} f(s,Z_s,\tilde{Z}_s) &= \gamma\rho_s^* b_s + \frac{\gamma(\gamma-1)}{2}|\rho_s^*\sigma_s|^2 + \frac{|Z_s|^2}{2} + \gamma\rho_s^*\sigma_s Z_s + (\gamma\rho_s^*\sigma_s + Z_s)q_s^* \\ &\quad + \psi_s^* \cdot ((1+\rho_s^*\beta_s)^{\gamma} e^{\tilde{Z}_s} - 1) + \int_{\mathbb{R}\setminus\{0\}} \Big[(1+\rho_s^*\beta_s(x))^{\gamma} e^{\tilde{Z}_s(x)} - \tilde{Z}_s(x) - \gamma\rho_s^*\beta_s(x) - 1\Big] n_p(dx), \end{aligned}$$

where $Z, \tilde{Z}$ belong to the solution of the BSDE (4.18). Now it is not hard to see that for any $\gamma < 0$ and $s \in [0,T]$, $(X_s^{(\rho)})^{\gamma}\mathrm{E}\left[\frac{dQ^{q,\psi}}{dP}|\mathcal{F}_s\right]$ is in L^1. Thus, since by definition for every s we have that $R_s^{\rho,(q,\psi)} = \frac{(X_s^{\rho})^{\gamma}}{\gamma}\mathrm{E}\left[\frac{dQ^{q,\psi}}{dP}|\mathcal{F}_s\right]\exp\{-Y_s\}$ with Y being bounded, we get that also $R_s^{\rho,(q,\psi)} \in L^1$ for every s. Therefore, by the definition of f in (4.17):

(a) The supermartingale condition in (iii) holds true, as for all ρ, the positive process $\tilde{A}^{\rho,(q^*,\psi^*)} := \exp(A^{\rho,(q^*,\psi^*)})$ is non-decreasing. (Note that $R_0 < 0$.)

(b) The submartingale condition in (iv) holds true, as for all (q,ψ), the process $\tilde{A}^{\rho^*,(q,\psi)}$ is non-increasing.

(c) The martingale condition in (v) holds true, as $\tilde{A}^{\rho^*,(q^*,\psi^*)} = 1$.

Hence, our family $R_s^{\rho,(q,\psi)}$ satisfies properties (i)-(v). The theorem is proved. □

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