Risk Apportionment: The Dual Story*

Louis R. Eeckhoudt
IESEG School of Management
CNRS-LEM UMR 9221
and CORE
Louis.Eeckhoudt@fucam.ac.be

Roger J. A. Laeven
Amsterdam School of Economics
University of Amsterdam, EURANDOM
and CentER
R.J.A.Laeven@uva.nl

Harris Schlesinger
Department of Finance
University of Alabama
and University of Konstanz
hschlesi@cba.ua.edu

This Version: January 19, 2018

Extended Online Version.

Abstract

By specifying model free preferences towards simple nested classes of lottery pairs, we develop the dual story to stand on equal footing with that of (primal) risk apportionment. The dual story provides an intuitive interpretation, and full characterization, of dual counterparts of such concepts as prudence and temperance. The direction of preference between these nested classes of lottery pairs is equivalent to signing the successive derivatives of the probability weighting function within Yaari's (1987) dual theory. We explore implications of our results for optimal portfolio choice and show that the sign of the third derivative of the probability weighting function may be naturally linked to a self-protection problem.

Keywords: Higher Order Risk Attitudes; Prudence; Temperance; Risk Apportionment; Non-Expected Utility Theory; Portfolio Choice; Self-Protection.

AMS 2010 Classification: Primary: 91B06, 91B16, 91B30; Secondary: 60E15, 62P05.

JEL Classification: D81, G11, G22.

*With great sadness, we lost our friend and co-author Harris Schlesinger, who passed away while we were in the process of writing this paper. We are very grateful for detailed comments from Sebastian Ebert, Johanna Etner, Christian Gollier, Glenn Harrison, Mike Hoy, Liqun Liu (discussant), Lisa Posey (discussant), Nicolas Treich, Michel Vellekoop, and Claudio Zoli. We are also grateful to conference and seminar participants at the EGRIE Meeting, the Risk Theory Society Meeting, the World Risk and Insurance Economics Congress, the Tinbergen Institute, and the KAFEE seminar at the University of Amsterdam for helpful comments. This paper was circulated earlier under the title "Prudence, temperance (and other virtues): The dual story". This research was funded in part by the Netherlands Organization for Scientific Research (Laeven) under grant NWO VIDI.

1 Introduction

Although first received with some skepticism, the notions of prudence and temperance have now been widely accepted almost on par with the fundamental concept of risk aversion, at least in an expected utility (EU) framework.

The expanding use of these notions, sometimes termed “higher order risk attitudes”, can be explained by the fact that they were progressively given a more general interpretation. Consider prudence, for instance. This term was coined by Kimball (1990) in an influential paper in which he showed that precautionary savings as an optimizing type of behavior is characterized in an EU framework by a positive third derivative of the utility function (i.e., “$U''' \geq 0$” or “prudence”). However, it is by now well-known that this positive sign of U''' can be justified more generally outside the specific decision problem of saving. This more primitive justification of prudence was initiated by Menezes, Geiss and Tressler (1980), who used the term “downside risk aversion”, and it was further pursued in Eeckhoudt and Schlesinger (2006), who also showed how to proceed from prudence to higher order risk attitudes. These authors first state a “model free” preference, namely that decision makers (DM’s) like to “combine good with bad” instead of having to face either everything good or everything bad. Next, this model free preference is shown to be translated into prudence ($U''' \geq 0$) within the EU model, and from prudence—by defining a sequence of nested lotteries and always asserting the preference for combining good with bad—the higher order risk attitudes may be obtained similarly, starting with temperance ($U'''' \leq 0$) at the fourth order.

It turns out besides that the simple primitive interpretation of prudence and higher order risk attitudes found in Eeckhoudt and Schlesinger (2006) lends itself easily to experimental verification. As a result, there is now an intensive experimental research activity around the concepts of prudence and temperance in an EU framework (e.g., Ebert and Wiesen (2011, 2014), Deck and Schlesinger (2010, 2014), and Noussair, Trautmann and van de Kuilen (2014), to name a few).

The preference for “combining good with bad” has appeared under different names in the literature. It was called “risk apportionment” in Eeckhoudt and Schlesinger (2006). A little earlier, Chiu (2005) referred to a "precedence relation", in which one stochastic dominant change precedes another. The phrase "combining good with bad" as a primitive trait first appeared in Eeckhoudt, Schlesinger and Tsetlin (2009).

While prudence, temperance, and higher order risk attitudes can be presented initially as natural properties in a model free environment, their interpretation and implementation have been developed so far exclusively within an EU framework.[1] In this paper, by specifying new model free preferences towards simple nested classes of lottery pairs, we develop the dual story to stand on equal footing with that of (primal) risk apportionment. The dual story provides an intuitive interpretation, and full characterization, of dual counterparts of such concepts as prudence, temperance, and other virtues. We show that the direction of preference between the nested classes of lottery pairs that we construct is equivalent to signing the m^{th} derivative of the probability weighting (or distortion) function within Yaari's (1987) dual theory (DT), with m an arbitrary positive integer.[2] It turns out that this development requires a fundamental departure from the approach of Eeckhoudt and Schlesinger (2006), which, as we will show, is unable to deliver the desired implications within the DT framework. The dual story we develop retains generic features of the primal story—e.g., a precedence relation—but crucially departs from it in its construction and implementation—e.g., by reference to what we will refer to as "squeezing" and "anti-squeezing" and to the "dual moments".

This paper thus represents a first step towards a more general interpretation of higher order risk attitudes within alternative non-EU decision models, such as rank-dependent utility and prospect theory (Kahneman and Tversky (1979), Quiggin (1982), Schmeidler (1986, 1989), Tversky and Kahneman (1992)), for which DT is a building block. Indeed, because DT is "orthogonal" to EU, our analysis not only reveals the differences with the primal story under EU, but it is also a prerequisite for a development of higher order risk attitudes as primitive

[1] In very recent work, Baillon (2017) generalizes these interpretations of prudence and higher order risk attitudes to a setting featuring ambiguity.

[2] One may therefore say that this paper constitutes the genuine appropriate dual counterpart of (primal) risk apportionment (Eeckhoudt and Schlesinger (2006)). Indeed, from a technical perspective, the contribution of the present paper to the extant literature on (dual or inverse) stochastic ordering in DT (e.g., Muliere and Scarsini (1989), Wang and Young (1998) and Chateauneuf, Gajdos and Wilthien (2002)) is similar to the contribution of Eeckhoudt and Schlesinger (2006) to the literature on (primal) stochastic ordering in EU (e.g., Whitmore (1970), Ekern (1980) and Menezes, Geiss and Tressler (1980)).

traits of behavior compatible with the more general models of choice under risk provided by rank-dependent utility and prospect theory. Based on our simple characterizations one may now develop tests on the signs of the higher order derivatives of the probability weighting function in these non-EU decision theories to be used in experimental analysis.

A positive sign of the third derivative of the probability weighting function is consistent with an "inverse S-shape", exhibited by the popular probability weighting functions proposed by Tversky and Kahneman (1992) (see also Wu and Gonzalez (1996, 1998)) and Prelec (1998) under typical parameter sets implied by experiments.[3] These inverse S-shaped probability weighting functions typically feature a positive sign for the odd derivatives and an alternating sign (first negative at low probability levels, then positive at high probability levels) for the even derivatives. Provided that the probability weighting function is first concave and then convex with second derivative equal to zero at the inflection point, a positive sign of the third derivative of the probability weighting function implies that the function becomes more concave when moving to the left of the inflection point and becomes more convex when moving to the right of the inflection point.

As is well-known, primal and dual stochastic dominance coincide up to the second order and may diverge from the third order onwards. As a by-product, which is of interest in its own right, the model free story appropriate for DT will nicely make apparent the fundamental reason behind this divergence.

Our results about the shape of the probability weighting function are relevant for the analysis of well-known problems in the economics of risk, such as portfolio choice and the level of self-protection in the presence of background risk. We first illustrate our results by deriving their implications for optimal portfolio choice with a risky asset, a risk-free asset and access to zero-mean financial derivative products on the risky asset. We show that, contrary to under EU (Gollier (1995)), an m^{th} order improvement of the risky asset's return, achieved by supplementing (hence, squeezing) the risky asset with an appropriate selection of derivative products (e.g., a straddle at the third order or a volatility spread at the fourth order), never

[3]The inverse S-shape reflects the psychological notion of diminishing sensitivity (in the domain of probabilities), which stipulates that DM's become less (more) sensitive to changes in the objective probabilities when they move away from (towards) the reference points 0 and 1.

reduces the demand for the risky asset under the DT model. Furthermore, we show that the third derivative of the probability weighting function naturally appears in a self-protection problem that trades off the risk of a loss and the effort of protecting against the loss, in the presence of an independent background risk. In particular, if the third derivative of the probability weighting function is positive ("dual prudence"), the background risk stimulates self-protection.

This paper is organized as follows. In Section 2, we fix the notation and setting, introduce some preliminaries for the DT decision model, and provide some basic intuition behind our results, by presenting simple numerical illustrations that make the links between the conventional model free preferences and the two models of choice under risk (EU versus DT) explicit. In Section 3, we introduce dual higher order risk attitudes by developing new model free preferences. Section 4 illustrates implications of our results for optimal portfolio choice. Section 5 shows that the sign of the third derivative of the probability weighting function is naturally linked to a self-protection problem. Section 6 contains the formal presentation of our general results. We conclude in Section 7 with a summary of the results and an indication of potential extensions. Proofs are relegated to the Appendix.

2 Preliminaries

2.1 Notation and Setting

We represent an n-state lottery A, assigning probabilities p_i to final wealth outcomes x_i, $i = 1, \ldots, n$, by $A = [\ x_1, p_1\ ;\ \ldots\ ;\ x_n, p_n\ ]$. We always assume that states are ordered according to their associated outcomes, from the lowest outcome state to the highest outcome state. Outcomes are assumed to be non-negative. That is, $0 \leq x_1 \leq \cdots \leq x_n$. With each lottery, generating a probability distribution over outcomes, one can associate a random variable. Henceforth, the lottery and its associated random variable are often identified. We write $\mathrm{P}[A \leq x] = F_A(x) = 1 - S_A(x)$, with F_A and S_A the cumulative and decumulative distribution functions of A, respectively. Furthermore, we denote by $\succeq$ a (weak) preference relation over lotteries.

Under Yaari's (1987) dual theory (DT), the evaluation V of the n-state lottery A is given by

$$V[A] = \int_0^\infty x \, \mathrm{d}h(F_A(x)) = \sum_{i=1}^{n} x_i \left(h(F_A(x_i)) - h(F_A(x_{i-1})) \right), \tag{2.1}$$

with $x_0 = 0$ and $F_A(x_0) = 0$ by convention, and with $h : [0,1] \to [0,1]$ satisfying $h(0) = 0$, $h(1) = 1$, and $h' \geq 0$, a probability weighting (or distortion) function, henceforth assumed to be differentiable for all degrees of differentiation on $(0,1)$.[4] We sometimes denote by $h^{(m)}$ the m^{th} derivative of h.[5]

We say that a lottery B dominates a lottery A in third-degree dual (or inverse) stochastic dominance order if

$$\mathrm{E}[A] \leq \mathrm{E}[B], \qquad \mathrm{E}[\min(A_1, A_2)] \leq \mathrm{E}[\min(B_1, B_2)], \quad \text{and} \quad {}^3F_A^{-1} \leq {}^3F_B^{-1}.$$

Here, A_1 (B_1) and A_2 (B_2) are two independent draws from lottery A (B). Furthermore, ${}^{m+1}F_A^{-1}(q) = \int_0^q {}^mF_A^{-1}(p)\, \mathrm{d}p$, $m = 1, 2, \ldots$, $0 \leq q \leq 1$, with ${}^1F_A^{-1} \equiv F_A^{-1}$, and inequalities between functions are understood pointwise. We refer to $\mathrm{E}[\min(A_1, A_2, \ldots, A_m)]$ as the m^{th} "dual moment" of A.[6] Similarly, we say that lottery B dominates lottery A in fourth-degree

[4] In the literature, following Yaari (1987), the evaluation under DT is often carried out by distorting the decumulative distribution function S_A instead of the cumulative distribution function F_A as in (2.1). Notice, however, that

$$V[A] = \int_0^\infty x \, \mathrm{d}h(F_A(x)) = \int_0^\infty x \, \mathrm{d}\left(1 - \bar{h}(S_A(x))\right) = \int_0^\infty \bar{h}(S_A(x)) \, \mathrm{d}x,$$

with $\bar{h}(p) := 1 - h(1-p)$, mapping the unit interval onto itself and satisfying $\bar{h}(0) = 0$, $\bar{h}(1) = 1$, and $\bar{h}' \geq 0$, and that positive signs of the higher order derivatives of $\bar{h}$ are equivalent to alternating signs of the higher order derivatives of h. In particular, concavity of h translates to convexity of $\bar{h}$. To be consistent with the alternating signs of the derivatives of the utility function in the EU model, we define the evaluation V under DT by distorting the cumulative distribution function F_A rather than the decumulative distribution function S_A. Thus, in our setting a concave probability weighting function h is equivalent to "strong" risk aversion in the sense of aversion to mean preserving spreads (Chew, Karni and Safra (1987) and Roëll (1987)), and, more generally, the higher order derivatives of the probability weighting function h will naturally alternate in sign, just like that of U, under a precedence relation. (A preference for combining good with good and bad with bad will turn out to be consistent with all derivatives of h having the same (positive) sign.)

[5] We use the notations h', h'', ... and $h^{(1)}$, $h^{(2)}$, ... interchangeably.

[6] In statistics, these moments are sometimes referred to as mean (first) order statistics. They measure the expected worst outcome in an experiment with repeated independent draws; see also David (1981). Primal

dual stochastic dominance order if

$$\mathrm{E}[A] \leq \mathrm{E}[B], \qquad \mathrm{E}[\min(A_1, A_2)] \leq \mathrm{E}[\min(B_1, B_2)],$$
$$\mathrm{E}[\min(A_1, A_2, A_3)] \leq \mathrm{E}[\min(B_1, B_2, B_3)], \quad \text{and} \quad {}^4F_A^{-1} \leq {}^4F_B^{-1}.$$

In full generality, we say that lottery B dominates lottery A in m^{th}-degree dual stochastic dominance order ($m = 2, 3, \ldots$) if

$$\mathrm{E}[A] \leq \mathrm{E}[B], \quad \mathrm{E}[\min(A_1, A_2)] \leq \mathrm{E}[\min(B_1, B_2)], \quad \ldots \quad ,$$
$$\mathrm{E}[\min(A_1, A_2, \ldots, A_{m-1})] \leq \mathrm{E}[\min(B_1, B_2, \ldots, B_{m-1})], \quad \text{and} \quad {}^mF_A^{-1} \leq {}^mF_B^{-1}.$$

Preferences for m^{th} order dual stochastic dominance, with first $(m-1)$ dual moments equal, are well-known to be linked to the sign of the m^{th} derivative of the probability weighting function within DT (Muliere and Scarsini (1989)), just like primal stochastic dominance orders are connected to the successive derivatives of the utility function under EU (Ekern (1980)). For further details on dual (inverse) stochastic dominance, we refer e.g., to De La Cal and Cárcamo (2010) and the references therein.

Under EU the sign of the m^{th} derivative of the utility function can be interpreted by comparing simple nested pairs of lotteries with equal $(m-1)$ primal moments, obtained via the apportionment of harms (Eeckhoudt and Schlesinger (2006)). To interpret the sign of the m^{th} derivative of the probability weighting function under DT, we develop simple nested lottery pairs that have equal $(m-1)$ dual moments.

---

moments occur under EU when considering power functions as utility functions. Similarly, dual moments occur under DT when considering power functions as probability weighting functions:

$$\begin{aligned}\mathrm{E}[\min(A_1, A_2, \ldots, A_m)] &= \int_0^\infty (1 - F_A(x))^m \,\mathrm{d}x = \int_0^\infty \bar{h}(1 - F_A(x)) \,\mathrm{d}x = \int_0^\infty x \,\mathrm{d}h(F_A(x)) \\ &= V[A],\end{aligned}$$

with $\bar{h}(p) = p^m$ and $\bar{h}(p) = 1 - h(1-p)$, $0 \leq p \leq 1$. Observe that the power probability weighting function satisfies $\bar{h}'' \geq 0$, hence $h'' \leq 0$.

2.2 Illustration and Intuition

Since it is well-known that EU and DT agree at the first and second orders in their evaluation of a sure reduction in wealth and a mean preserving spread (see Yaari (1986), Chew, Karni and Safra (1987), Roëll (1987) and Muliere and Scarsini (1989)) we develop here numerical examples to illustrate that they may, but need not, diverge at the third order. These examples build the intuition behind the reason why EU and DT may diverge at the third order and motivate the adjustments that have to be made to the story of "combining good with bad" used to interpret the signs of the successive derivatives of the utility function under EU. While it is known that third order primal and dual stochastic dominance do not in general coincide (see Muliere and Scarsini (1989) and its references), we illustrate their potential divergence here in the context of (primal) risk apportionment (Eeckhoudt and Schlesinger (2006)).

To motivate their paper in which they interpret the sign of U''', Menezes, Geiss and Tressler (1980) refer to Mao's lotteries[7] given by

$$A = [\ 0, 1/4\ ;\ 2, 3/4\ ], \qquad\qquad B = [\ 1, 3/4\ ;\ 3, 1/4\ ].$$

Note that A and B have the first two moments in common, that is, have the same mean and variance. Lotteries A and B can be (viewed as) generated from a common initial lottery I given by

$$I = [\ 1, 1/2\ ;\ 2, 1/2\ ],$$

to which an independent zero-mean risk $\tilde{\varepsilon}$ given by

$$\tilde{\varepsilon} = [\ -1, 1/2\ ;\ 1, 1/2\ ],$$

is "added". If $\tilde{\varepsilon}$ is added to the worst state of I (i.e., that with outcome 1) one generates A:

$$A = [\ 1 + \tilde{\varepsilon}, 1/2\ ;\ 2 + 0, 1/2\ ],$$

[7] This name originates from the fact that these lotteries were used by James Mao (1970) in an experiment with business men.

while if $\tilde{\varepsilon}$ is added to the best state of I (i.e., that with outcome 2) one generates B:

$$B = [\ 1 + 0, 1/2\ ;\ 2 + \tilde{\varepsilon}, 1/2\ ].$$

Observe that B is obtained from I by adding the bad risk $\tilde{\varepsilon}$ (bad since $\tilde{\varepsilon}$ is second order stochastically dominated by 0) to the good state of I while the converse is true for A. One might alternatively say that in A the bad ($\tilde{\varepsilon}$) precedes the good (0). In B, on the contrary, the good (0) precedes the bad ($\tilde{\varepsilon}$).[8] Economic agents who like to "combine good with bad" prefer B to A. Under EU, this corresponds to $U''' \geq 0$. See Eeckhoudt and Schlesinger (2006) for a precise statement of this result.

Now consider DT instead of EU. What do we know for DT? Note first that A and B also have the first two *dual* moments in common, where the second dual moments amount to:

$$\mathrm{E}\left[\min(A_1, A_2)\right] = (1 - (3/4)^2)0 + (3/4)^2 2 = 9/8.$$
$$\mathrm{E}\left[\min(B_1, B_2)\right] = (1 - (1/4)^2)1 + (1/4)^2 3 = 9/8.$$

Furthermore, one easily computes that

$$V[A] = h(1/4)0 + (1 - h(1/4))2, \qquad V[B] = h(3/4)1 + (1 - h(3/4))3,$$

with $h(0) = 0$, $h(1) = 1$, $h' \geq 0$, and where we assume $h'' \leq 0$. We now have the following result:

Proposition 2.1 *If h is quadratic, then $V[A] = V[B]$.*

Proposition 2.1 says that, under DT, the DM is indifferent between Mao's lotteries if the distortion function is quadratic. It suggests that, for this specific pair of lotteries, preferences are dictated by the behavior of higher order (> 2) derivatives of the distortion function. This is indeed confirmed by the next result (the formal general statement of which is deferred to Section 6 — see Theorem 6.1):

[8] This alternative presentation will be useful to understand the dual story explicated in the next section.

Proposition 2.2 *If $h''' \geq (\leq)\ 0$, then $V[A] \leq (\geq)\ V[B]$, that is, B is preferred (dispreferred) to A.*

Since EU and DT agree at the third order in their evaluations of Mao's lotteries (under EU $U''' \geq 0 \Rightarrow B \succeq A$ and under DT $h''' \geq 0 \Rightarrow B \succeq A$), one may wonder whether this result extends to all lottery pairs defined by Eeckhoudt and Schlesinger (2006) at the third order (see, specifically, Definition 1 on p. 282 of Eeckhoudt and Schlesinger (2006)). By providing two numerical examples, we show that EU and DT may, but need not, agree for these Eeckhoudt and Schlesinger pairs of lotteries.[9]

First, consider an initial lottery $\tilde{I}$ given by

$$\tilde{I} = [\ 4, 1/2\ ;\ 10, 1/2\ ],$$

and then add to one of the states the independent zero-mean risk $\tilde{\zeta}$ given by

$$\tilde{\zeta} = [\ -2, 1/2\ ;\ 2, 1/2\ ].$$

This generates either $\tilde{A}$ or $\tilde{B}$ given by

$$\tilde{A} = [\ 2, 1/4\ ;\ 6, 1/4\ ;\ 10, 1/2\ ], \qquad \tilde{B} = [\ 4, 1/2\ ;\ 8, 1/4\ ;\ 12, 1/4\ ]. \tag{2.2}$$

Clearly, $\tilde{A}$ and $\tilde{B}$ again have the first two moments in common, and it is well-known that under EU,

$$U''' \geq (\leq)\ 0 \Rightarrow \tilde{B} \succeq (\preceq) \tilde{A}.$$

Now let us consider the comparison between $\tilde{A}$ and $\tilde{B}$ under DT. One easily verifies that $\tilde{A}$ and $\tilde{B}$ have the same mean and the same dual moments $\mathrm{E}\left[\min(\tilde{A}_1, \tilde{A}_2)\right]$ and $\mathrm{E}\left[\min(\tilde{B}_1, \tilde{B}_2)\right]$,

[9] It was sometimes conjectured — falsely, as we know now — that the agreement of EU and DT at the third order is true only for Mao's lotteries.

which amount to

$$\mathrm{E}\left[\min(\tilde{A}_1, \tilde{A}_2)\right] = (7/16)2 + (5/16)6 + (4/16)10 = 21/4.$$
$$\mathrm{E}\left[\min(\tilde{B}_1, \tilde{B}_2)\right] = (12/16)4 + (3/16)8 + (1/16)12 = 21/4.$$

We have the following result:

Proposition 2.3 *If h is quadratic, then $V[\tilde{A}] = V[\tilde{B}]$.*

Besides, following the results developed in Section 6 (see Theorem 6.1), we find that:

Proposition 2.4 *If $h''' \geq (\leq)\ 0$, then $V[\tilde{A}] \leq (\geq)\ V[\tilde{B}]$, that is, $\tilde{B}$ is preferred (dispreferred) to $\tilde{A}$.*

It thus turns out that for some lotteries used by Eeckhoudt and Schlesinger (2006), EU and DT produce the same ranking at the third order. However, we now show by means of another example that this is not always the case. This "counterexample" serves to undermine the false impression given by Mao's and, in fact, various other lotteries in the class of lotteries used by Eeckhoudt and Schlesinger (2006), such as (2.2), that EU and DT agree at the third order. This divergence is, in principle, known but shown here explicitly for the Eeckhoudt and Schlesinger (2006) construction of apportioning a zero-mean risk.

Indeed, start from an initial lottery $\hat{I}$ given by

$$\hat{I} = [\ 2, 1/2\ ;\ 3, 1/2\ ],$$

and add the independent zero-mean risk $\tilde{\theta}$ given by

$$\tilde{\theta} = [\ -2, 1/3\ ;\ 1, 2/3\ ],$$

to one of the states of $\hat{I}$. Note that this zero-mean risk $\tilde{\theta}$ is very different from the previous ones we used $(\tilde{\varepsilon}, \tilde{\zeta})$: it is no longer symmetric and besides it will induce a change in the ranking of the states. Indeed, the new lotteries, after the apportionment of $\tilde{\theta}$, are $\hat{A}$ and $\hat{B}$ given by

$$\hat{A} = [\ 0, 1/6\ ;\ 3, 5/6\ ], \qquad\qquad \hat{B} = [\ 1, 1/6\ ;\ 2, 3/6\ ;\ 4, 2/6\ ].$$

From Eeckhoudt and Schlesinger (2006), we know that under EU,

$$U''' \geq (\leq)\ 0 \Rightarrow \hat{B} \succeq (\preceq)\ \hat{A},$$

and note that $\mathrm{E}[\hat{A}] = \mathrm{E}[\hat{B}]$, $\mathrm{Var}[\hat{A}] = \mathrm{Var}[\hat{B}]$, while $\mathrm{Skew}[\hat{A}] < \mathrm{Skew}[\hat{B}]$.

If we turn to DT and consider a quadratic distortion function so that $h''' = 0$, it can be shown that $\hat{A} \succeq \hat{B}$, producing a ranking at the third order different from EU: the EU DM is indifferent between $\hat{A}$ and $\hat{B}$ whenever $U''' = 0$. Indeed, with a probability weighting function that satisfies the same properties as in the proof of Proposition 2.1,

$$\begin{aligned} V[\hat{A}] &= ((1/6)(1+\beta) - (1/36)\beta)0 + (1 - ((1/6)(1+\beta) - (1/36)\beta))3 \\ &= 5/2 - (5/12)\beta. \\ V[\hat{B}] &= ((1/6)(1+\beta) - (1/36)\beta)1 \\ &\quad + (((4/6)(1+\beta) - (16/36)\beta) - ((1/6)(1+\beta) - (1/36)\beta))2 \\ &\quad + (1 - ((4/6)(1+\beta) - (16/36)\beta))4 \\ &= 5/2 - (7/12)\beta. \end{aligned}$$

Hence, $V[\hat{A}] - V[\hat{B}] = (1/6)\beta \geq 0$ (since $0 \leq \beta \leq 1$), with $V[\hat{A}] - V[\hat{B}]$ strictly larger than zero whenever $\beta > 0$ (or $h'' < 0$). The basic reason for this result is the fact that the second dual moments of $\hat{A}$ and $\hat{B}$ differ.

It thus appears that the model free prescription that favors "combining good with bad" in the particular way as suggested by Eeckhoudt and Schlesinger (2006) leads to an interpretation of the signs of the successive derivatives of U within the EU model; however, this model free principle does not always generate a similar interpretation for the signs of the successive derivatives of the probability weighting function under the DT framework. Thus, if one desires to obtain for DT and the signs of the successive derivatives of the probability weighting function a development that parallels that within EU, one has to modify the initial model free preferences. This is the purpose of the next section.

3 Model Free Preferences

To provide an intuitive interpretation to the signs of the successive derivatives of the probability weighting function beyond the second order[10] we now develop the appropriate model free preferences. We will continue to assert that DM's want to combine good with bad. In Chiu's (2005) equivalent terminology, the DM still satisfies a precedence relation. He favors that the good precedes the bad. However, our definition of good and bad will be based on the concepts of "squeezing" and "anti-squeezing" a distribution instead of adding zero-mean risks and zero with certainty as in Eeckhoudt and Schlesinger (2006). Squeezing and anti-squeezing will serve as the main building blocks in our approach. The sequence of squeezes and anti-squeezes that we develop will preserve dual rather than primal moments.

Squeezing occurs when we transform an initial lottery

$$L = [\ x_1, p_1\ ;\ \ldots\ ;\ x_i, p_i + p\ ;\ \ldots\ ;\ x_j, p_j + p\ ;\ \ldots\ ;\ x_n, p_n\ ],$$

with $p_k \geq 0$ (where equality to zero is explicitly permitted), $k = 1, \ldots, n$, and $p > 0$, into a lottery D given by

$$D = [\ x_1, p_1\ ;\ \ldots\ ;\ x_i, p_i\ ;\ x_i + x, p\ ;\ \ldots\ ;\ x_j - x, p\ ;\ x_j, p_j\ ;\ \ldots\ ;\ x_n, p_n\ ],$$

with $x > 0$. Anti-squeezing occurs when x is replaced by $-x$, that is, when L is transformed into a lottery C given by

$$C = [\ x_1, p_1\ ;\ \ldots\ ;\ x_i - x, p_i\ ;\ x_i, p\ ;\ \ldots\ ;\ x_j, p\ ;\ x_j + x, p_j\ ;\ \ldots\ ;\ x_n, p_n\ ].$$

[10]As is well-known, both $U'' \leq 0$ and $h'' \leq 0$ correspond to strong risk aversion (i.e., aversion to mean preserving spreads); see the references in Section 2.2.

Clearly, both squeezing and anti-squeezing preserve the mean. Note the generality of these transformations. Depending on the specification of x and p, a squeeze and an anti-squeeze can be interpreted both as a shift in outcomes and as a shift in probabilities.

In this section, we explicate the dual story in examples. Section 6 contains the general approach. We first return to the second order and consider an initial lottery $L^{(2)}$ given by[11]

$$L^{(2)} = [\ 1, 1/2\ ;\ 2, 1/2\ ].$$

Now we transform the lottery $L^{(2)}$ into a lottery $D^{(2)}$ by squeezing. Specifically, we "attach" (state-wise with the state probabilities matched)

$$\mathcal{G}^{(2)} = [\ x, p\ ] \quad \text{and} \quad \mathcal{B}^{(2)} = [\ -x, p\ ],$$

to $L^{(2)}$, where in this case we let $p = 1/n$, $n = 2$, and where $x = 1/M$, $M \geq 2$.[12] We assume the good ($\mathcal{G}^{(2)}$) precedes the bad ($\mathcal{B}^{(2)}$). This squeeze yields the new lottery $D^{(2)}$ given by

$$D^{(2)} = [\ 1 + 1/M, 1/2\ ;\ 2 - 1/M, 1/2\ ].$$

Of course, an anti-squeeze of $L^{(2)}$ that is same-sized but opposite to the squeeze, achieved by attaching $\mathcal{B}^{(2)}$ and $\mathcal{G}^{(2)}$ state-wise with the bad now preceding the good, would generate $C^{(2)}$ given by

$$C^{(2)} = [\ 1 - 1/M, 1/2\ ;\ 2 + 1/M, 1/2\ ].$$

See the illustration in Figure 1.

Clearly, under strong risk aversion, $D^{(2)} \succeq C^{(2)}$.[13] These preferences correspond to $h'' \leq 0$ under DT and to $U'' \leq 0$ under the EU model. Indeed, squeezing and anti-squeezing are

[11] The superscript $^{(2)}$ refers to "second order" and more generally the superscript $^{(m)}$ refers to "m^{th} order".

[12] The condition on M guarantees that the squeeze does not change the initial ranking of outcomes.

[13] For reasons that become apparent in Section 6 we don't compare the changed lotteries $C^{(2)}$ and $D^{(2)}$ to the initial lottery $L^{(2)}$ they are generated from, although this comparison is straightforward in this particular case, in which $D^{(2)} \succeq L^{(2)} \succeq C^{(2)}$. In fact, throughout this section, $D^{(m)} \succeq L^{(m)} \succeq C^{(m)}$, $m = 2, 3, 4$, for DT DM's having probability weighting functions with higher order derivatives that alternate in sign. For brevity, we sometimes directly construct $D^{(m)}$ from $C^{(m)}$ in the following sections.

Figure 1: Squeezing and Anti-Squeezing at the Second Order
This figure plots the transformation from $L^{(2)}$ to $D^{(2)}$ (left panel) and from $L^{(2)}$ to $C^{(2)}$ (right panel), with $x_1 = 1$, $x_2 = 2$, and $p = 1/2$. $\mathcal{G}^{(3)}$ and $\mathcal{B}^{(3)}$ will be used to develop the dual story at the third order.

(a.) Second order: Good precedes bad (b.) Second order: Bad precedes good

special cases of a mean preserving contraction and a mean preserving spread in the sense of Rothschild and Stiglitz (1970), and DT and EU are known to agree in their evaluation of a mean preserving contraction and a mean preserving spread.

Turning now to the third order, the agreement between DT and EU may collapse, as already anticipated in Section 2.2. The model free preferences based on the precedence relation towards good and bad coupled with the notions of squeezing and anti-squeezing are deployed as follows.[14] We start from an initial lottery $L^{(3)}$ given by

$$L^{(3)} = [\ 1, 1/3\ ;\ 2, 1/3\ ;\ 4, 1/3\ ].$$

Then we generate $D^{(3)}$ as

$$D^{(3)} = [\ 1 + 1/M, 1/3\ ;\ 2 - 2/M, 1/3\ ;\ 4 + 1/M, 1/3\ ],$$

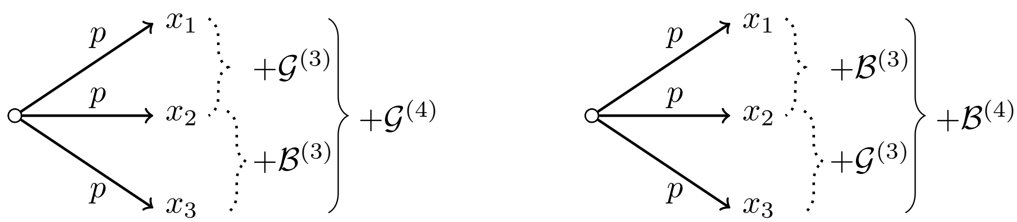
with $M \geq 3$. To obtain $D^{(3)}$ from $L^{(3)}$, we first transform the worst two states of $L^{(3)}$ by applying a squeeze, attaching

$$\mathcal{G}^{(3)} = [\ x, p\ ;\ -x, p\ ],$$

where we take in this case $p = 1/n$, $n = 3$, and $x = 1/M$, $M \geq 3$. Then on the best two states

[14]This is again an illustration. The general treatment is contained in Section 6.

Figure 2: Squeezing and Anti-Squeezing Sequences at the Third Order
This figure plots the transformation from $L^{(3)}$ to $D^{(3)}$ (left panel) and from $L^{(3)}$ to $C^{(3)}$ (right panel), with $x_1 = 1$, $x_2 = 2$, $x_3 = 4$, and $p = 1/3$. $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ will be used to develop the dual story at the fourth order.



(a.) Third order: Good precedes bad (b.) Third order: Bad precedes good

of $L^{(3)}$ we perform an anti-squeeze, achieved by attaching

$$\mathcal{B}^{(3)} = [\ -x, p\ ;\ x, p\ ],$$

with $p = 1/n$, $n = 3$, and $x = 1/M$, $M \geq 3$. In the transformation from $L^{(3)}$ to $D^{(3)}$, the good ($\mathcal{G}^{(3)}$) precedes the bad ($\mathcal{B}^{(3)}$). To keep the required number of states as small as possible, for ease and parsimony of exposition, we consider in this illustration a lottery $L^{(3)}$ with only 3 states, having moreover equal probabilities of occurrence, and we squeeze and anti-squeeze at an overlapping state (that with outcome 2) and by the same amount. These restrictions are, however, not required for our general approach; see Section 6 for further details.

Similarly, we generate a lottery $C^{(3)}$ given by

$$C^{(3)} = [\ 1 - 1/M, 1/3\ ;\ 2 + 2/M, 1/3\ ;\ 4 - 1/M, 1/3\ ],$$

obtained from the initial lottery $L^{(3)}$ by letting the bad, $\mathcal{B}^{(3)}$, precede the good, $\mathcal{G}^{(3)}$. See the illustration in Figure 2.

As shown in Section 6, to which a precise statement of this result is deferred—see Theorems 6.1 and 6.2—, "dual prudence" (or "$h''' \geq\ 0$") corresponds to "$D^{(3)} \succeq C^{(3)}$". Both the well-known probability weighting function of Tversky and Kahneman (1992) and that of Prelec (1998) exhibit $h''' \geq 0$ under the typical parameter sets implied by experiments.

It is very important to realize that there is no unanimity among EU maximizers with $U''' \geq 0$ on the comparison between $C^{(3)}$ and $D^{(3)}$. This is essentially so because these two lotteries with the same mean have different variances and skewnesses. Indeed, $\text{Var}[D^{(3)}] > \text{Var}[C^{(3)}]$ and $\text{Skew}[D^{(3)}] > \text{Skew}[C^{(3)}]$. As a result, EU DM's who are relatively more (less) risk averse than prudent prefer $C^{(3)}$ ($D^{(3)}$). While our story of squeezing and anti-squeezing may yield different primal moments (variances in particular) for the lotteries that are compared, it preserves equality of the second dual moments. Indeed, one may verify that

$$\mathrm{E}\left[\min(D_1^{(3)}, D_2^{(3)})\right] = \mathrm{E}\left[\min(C_1^{(3)}, C_2^{(3)})\right].$$

This is the fundamental reason why EU and DT may diverge from the third order onwards.

We note that if we would have started from an initial lottery $\tilde{L}^{(3)}$ given by

$$\tilde{L}^{(3)} = [\ 1, 1/3\ ;\ 2, 1/3\ ;\ 3, 1/3\ ],$$

equality of the variances would have been preserved in the transformation to the corresponding $\tilde{D}^{(3)}$ and $\tilde{C}^{(3)}$. In this case both EU DM's with $U''' \geq 0$ and DT DM's with $h''' \geq 0$ would have preferred $\tilde{D}^{(3)}$ over $\tilde{C}^{(3)}$. That is, EU and DT may, but need not, diverge at the third order.

In order to interpret the sign of the fourth derivative of h we start from an expanded lottery $L^{(4)}$ given by

$$L^{(4)} = [\ 1, 1/4\ ;\ 2, 1/4\ ;\ 4, 1/4\ ;\ 7, 1/4\ ].$$

Now on the worst three states of $L^{(4)}$ we conduct the beneficial transformation described at the third order (i.e., that with $\mathcal{G}^{(3)}$ preceding $\mathcal{B}^{(3)}$), attaching

$$\mathcal{G}^{(4)} = [\ x, p\ ;\ -2x, p\ ;\ x, p\ ],$$

with in this case $p = 1/n$, $n = 4$, and $x = 1/M$, $M \geq 4$. Next, on the best three states of $L^{(4)}$

we conduct exactly the opposite transformation, by attaching

$$\mathcal{B}^{(4)} = [\ -x, p\ ;\ 2x, p\ ;\ -x, p\ ],$$

so that we obtain a lottery $D^{(4)}$ given by

$$D^{(4)} = [\ 1 + 1/M, 1/4\ ;\ 2 - 3/M, 1/4\ ;\ 4 + 3/M, 1/4\ ;\ 7 - 1/M, 1/4\ ].$$

Conversely, by letting the bad precede the good we generate

$$C^{(4)} = [\ 1 - 1/M, 1/4\ ;\ 2 + 3/M, 1/4\ ;\ 4 - 3/M, 1/4\ ;\ 7 + 1/M, 1/4\ ].$$

In Section 6, we show that $D^{(4)}$ is unanimously preferred over $C^{(4)}$ by DT DM's with "dual temperance" (or "$h'''' \leq 0$")—see Theorems 6.3 and 6.4.

To see the implications of the dual story within the EU model, it is convenient to compare $D^{(4)}$ to $C^{(4)}$ when $M = 4$. While in this case $C^{(4)}$ and $D^{(4)}$ have the same mean and variance, one easily verifies that $D^{(4)}$ has both a smaller skewness and a smaller kurtosis than $C^{(4)}$. Hence, there cannot be unanimity among the prudent and temperate EU DM's about the appreciation of the two lotteries: some will prefer $D^{(4)}$ while others will prefer $C^{(4)}$, depending upon their relative degrees of prudence and temperance. This observation makes explicit the reason why at the fourth order EU and DT may (continue to) diverge: while the sequence of squeezing and anti-squeezing at the fourth order may produce different third primal moments for $C^{(4)}$ and $D^{(4)}$, it preserves the equality of the third dual moments, corroborating again their relevance for the dual story.

We finally note that the lotteries at the third and fourth orders are generated by simple iterations of the transformations relevant at the second and third orders, respectively. Hence, this analysis can be pursued up to arbitrary order to interpret the signs of the successive derivatives of the probability weighting function. We thus obtain, as in the Eeckhoudt and Schlesinger (2006) approach, a sequence of simple nested lotteries that yield now the appropriate interpretation of the signs of $h^{(m)}$.

4 Portfolio Choice with Derivatives

Consider a DT investor with initial sure wealth w_0. Suppose that he allocates an amount α to a risky asset (the "stock") and an amount $w_0 - \alpha$ to a risk-free asset (the "bond"). The bond (stock) earns a sure (risky) return of r (R) per unit invested. Assume that r and R are independent of the amounts invested. The investor aims to determine the optimal amount α^*. Because $(w_0-\alpha)(1+r)+\alpha(1+R) = w_0(1+r)+\alpha(R-r)$ and the DT evaluation is translation invariant and positively homogeneous, his problem reads:

$$\arg\max_\alpha\{\alpha(V[R]-r)\}. \tag{4.1}$$

We add the constraint $0 \leq \alpha \leq w_0$. It readily follows that the optimal solution α^* is a corner solution.

Suppose $V[R] < r$. Then the optimal solution is $\alpha^* = 0$. In this case, the investor fully invests in the bond. Now imagine that the investor is offered an improvement to the distribution of the risky return. In particular, he is offered the possibility of supplementing the stock with derivative products on the stock that have zero expected value at zero cost. The derivative products are selected by applying the dual story such that they induce, through squeezing and anti-squeezing, an m^{th} order improvement of R, with $m \geq 2$. The return of the risky portfolio of stock and derivatives jointly is denoted by $\bar{R}$. (We provide details on the derivative products shortly.)

Then two cases are possible: (i) If $V[\bar{R}] \leq r$, full investment in the bond remains optimal. (ii) If the improvement is sufficiently large so that $V[\bar{R}] > r$, the DT investor will shift from one corner solution (full investment in the bond, i.e., $\alpha^* = 0$) to the other corner solution (full investment in the risky portfolio, i.e., $\alpha^* = w_0$).

To illustrate these m^{th} order improvements, suppose for ease of exposition that $r \equiv 0$. Consider the second order first. Assume that the stock price takes the values 1 and 3 each with probability 1/2. Invoking the dual story, we find that a long position in a put option combined with a short position in a call option, each with a strike price of 2 such that the joint expected value is zero, improves the attractiveness of the risky portfolio ($\bar{R}$ versus R) whenever $h'' \leq 0$. Indeed, the difference between $\bar{R}$ and R can in this case be seen as the result of squeezing and anti-squeezing.[15] As visualized in the top left panel of Figure 3, this combination of long put and short call provides a hedge against adverse stock scenarios, which is financed by giving up upward potential.

[15] Throughout this section the risky stock price $S_0(1+R)$, with S_0 the initial stock price, plays the role of the lotteries $C^{(m)}$ encountered in the dual story, while the stock-and-derivatives portfolio price $S_0(1+\bar{R})$ plays the role of the lotteries $D^{(m)}$. The derivative supplements directly generate the improvement $\bar{R}$ from R. Specifically, $R = (C^{(m)} - S_0)/S_0$ and $\bar{R} = (D^{(m)} - S_0)/S_0$. At the second order,

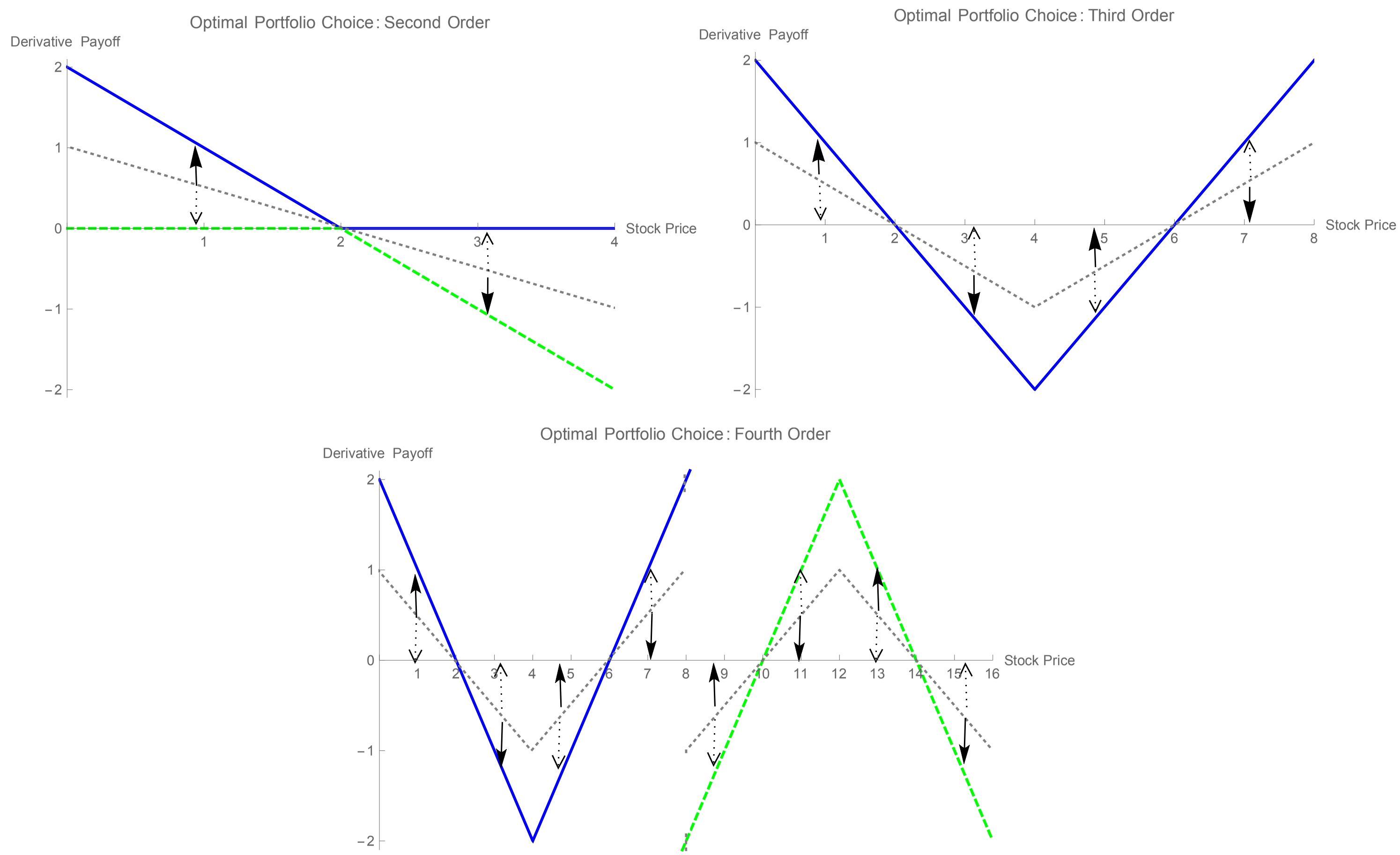
$$C^{(2)} = [\ 1, 1/2\ ;\ 3, 1/2\ ] \qquad \text{and} \qquad D^{(2)} = [\ 2, 1/2\ ;\ 2, 1/2\ ].$$

Of course, $C^{(2)}$ and $D^{(2)}$ may be generated from a common initial lottery $L^{(2)} = [\ 3/2, 1/2\ ;\ 5/2, 1/2\ ]$ by squeezing and anti-squeezing, that is, by attaching

$$\mathcal{G}^{(2)} = [\ 1/2, 1/2\ ] \qquad \text{and} \qquad \mathcal{B}^{(2)} = [\ -1/2, 1/2\ ],$$

to $L^{(2)}$. If the good (bad) precedes the bad (good), $D^{(2)}$ ($C^{(2)}$) is generated.

Figure 3: Optimal Portfolio Choice with Derivatives.
This figure plots the payoff functions of the derivative products to supplement the stock inducing an m^{th} order improvement, $m = 2, 3, 4$. At each order, the derivative products have zero (joint) expected value. At the second order (top left panel), we supplement the stock with a long position in a put option (blue, solid) and a short position in a call option (green, dashed), both with a strike price of 2. At the third order (top right panel), we enter into a straddle option (blue) at stock price 4. Finally, at the fourth order (bottom panel), we add a long straddle at stock price 4 (blue) and a short straddle at stock price 12 (green, dashed). Both the stock price itself and the stock-and-derivatives portfolio price can at each order be generated from a common initial lottery (in the form of a stock-and-derivatives portfolio) represented by the grey dotted lines through (a sequence of) squeezes and anti-squeezes represented by the solid and dashed arrows, respectively, according to the dual story; see also footnotes 15 and 16.



Next, consider the third order. Assume now that the stock price takes the values $\{1, 3, 5, 7\}$ each with probability 1/4. Then, deploying the dual story, one may readily verify that a so-called "straddle" option at stock price 4 such that the expected value is zero, improves the attractiveness of the risky portfolio ($\bar{R}$ versus R) whenever $h''' \geq 0$. As visualized in the top right panel of Figure 3, the straddle pays off in bad stock scenarios and in good stock scenarios, but generates losses in intermediate stock scenarios.[16] Because the straddle increases not only the skewness but also the variance of the risky portfolio, there would be no unanimity for the straddle supplement under EU, contrary to under DT.

Finally, at the fourth order, assume the stock price takes the values $\{1, 3, 5, 7, 9, 11, 13, 15\}$ each with probability 1/8. Then, the dual story implies that a long straddle at stock price 4 combined with a short straddle at stock price 12, with joint expected value equal zero, improves the attractiveness of the risky portfolio provided $h'''' \leq 0$. This combination of a long and short straddle, visualized in the last panel of Figure 3, is a popular and simple case of a so-called "volatility spread".[17]

We thus find that an m^{th} order (dual) improvement of the risky asset's return distribution never reduces the demand for the risky asset when the successive derivatives of the probability weighting function alternate in sign. This stands in sharp contrast to the familiar results under EU, where an m^{th} order (primal) improvement of R has ambiguous demand effects, even for $m \geq 1$. Indeed, to obtain the natural result that such improvements induce a higher demand for the risky asset under EU, additional non-trivial conditions need to be imposed (see Section 9.3 of Eeckhoudt and Gollier (1995) and Gollier (1995) for details).[18]

---

[16] At the third order, $R = (C^{(3)} - S_0)/S_0$ and $\bar{R} = (D^{(3)} - S_0)/S_0$, where

$$C^{(3)} = [\ 1, 1/4\ ;\ 3, 1/4\ ;\ 5, 1/4\ ;\ 7, 1/4\ ] \qquad \text{and} \qquad D^{(3)} = [\ 2, 1/4\ ;\ 2, 1/4\ ;\ 4, 1/4\ ;\ 8, 1/4\ ].$$

They can be generated from an initial lottery $L^{(3)} = [\ 3/2, 1/4\ ;\ 5/2, 1/4\ ;\ 9/2, 1/4\ ;\ 15/2, 1/4\ ]$ by attaching

$$\mathcal{G}^{(3)} = [\ 1/2, 1/4\ ;\ -1/2, 1/4\ ] \qquad \text{and} \qquad \mathcal{B}^{(3)} = [\ -1/2, 1/4\ ;\ 1/2, 1/4\ ],$$

at non-overlapping states.

[17] The payoffs of the long and short straddle are digitally set to 0 at stock price 8 for simplicity.

[18] See also the analysis of Dittmar (2002) in the EU model.

5 Self-Protection with Background Risk

In this section, we consider a DT DM with initial sure wealth w_0 who faces the risk of losing the monetary amount $\ell > 0$. The probability of occurrence of the loss, p, depends on the self-protection[19] effort, e, exerted by the DM. In particular, $p(e)$ is decreasing in e. Effort is measured in monetary equivalents. The DM aims to determine the optimal level of effort e^* that maximizes his DT evaluation.

We analyze this problem in the presence of an independent background risk. This background risk may e.g., be due to the risk of holding risky financial assets or to uncertain labor income. The sign of h''' appears to play an essential role in this self-protection problem with background risk. Hence, while under EU the a priori validity of $U''' \geq 0$ is connected to a savings decision (Kimball (1990)), we show that under DT the a priori validity of $h''' \geq 0$ may be linked to a self-protection problem.

With an independent binary background risk taking the value $\pm\varepsilon$, $\varepsilon > 0$, each with probability 1/2, the self-protection problem can be represented by the following 4-state lottery $S(e)$:

$$\begin{aligned} S(e) = [\ & w_0 - \ell - \varepsilon - e, p(e)/2\ ;\ w_0 - \ell + \varepsilon - e, p(e)/2\ ; \\ & w_0 - \varepsilon - e, (1-p(e))/2\ ;\ w_0 + \varepsilon - e, (1-p(e))/2\ ], \end{aligned}$$

assuming $2\varepsilon < \ell$. If $2\varepsilon > \ell$, the middle two states of S change places. The DT DM solves:

$$\arg\max_e \{V\,[S(e)]\}. \tag{5.1}$$

We assume throughout that V is concave in e.

[19] Adopting the terminology of Ehrlich and Becker (1972).

If $2\varepsilon < \ell$,

$$\begin{aligned}V[S(e)] = h(p(e)/2)(w_0 - \ell - \varepsilon - e) + (h(p(e)) - h(p(e)/2))(w_0 - \ell + \varepsilon - e)\\ + (h((1+p(e))/2) - h(p(e)))(w_0 - \varepsilon - e)\\ + (1 - h((1+p(e))/2))(w_0 + \varepsilon - e),\end{aligned}$$

and the first-order condition for optimality reads

$$\begin{aligned}\frac{V[S(e)]}{\mathrm{d}e} &= (p'(e)/2)h'(p(e)/2)\left((w_0 - \ell - \varepsilon - e) - (w_0 - \ell + \varepsilon - e)\right)\\ &\quad + p'(e)h'(p(e))\left((w_0 - \ell + \varepsilon - e) - (w_0 - \varepsilon - e)\right)\\ &\quad + (p'(e)/2)h'((1+p(e))/2)\left((w_0 - \varepsilon - e) - (w_0 + \varepsilon - e)\right)\\ &\quad - 1 = 0.\end{aligned}$$

After obvious simplifications, we obtain

$$p'(e)\varepsilon\left(-h'(p(e)/2) + 2h'(p(e)) - h'((1+p(e))/2)\right) - p'(e)h'(p(e))\ell - 1 = 0. \tag{5.2}$$

We note that in the absence of any background risk, i.e., when $\varepsilon = 0$, the first-order condition reduces to

$$-p'(e)h'(p(e))\ell - 1 = 0. \tag{5.3}$$

Now assume first, like in Eeckhoudt and Gollier (2005) for EU, that $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) = 1/2$. This means that the DT DM optimally selects an effort level e such that the probability of the occurrence of a loss is 1/2, in the absence of background risk. Under this assumption, we find from (5.2) that the impact of the background risk is linked to the sign of

$$-h'(1/4) + 2h'(1/2) - h'(3/4) = \left(h'(1/2) - h'(1/4)\right) - \left(h'(3/4) - h'(1/2)\right). \tag{5.4}$$

If the sign of h''' is positive, then (5.4) is negative, and note that $p'(e)\varepsilon$ is negative, too. Therefore, and in view of the concavity of V with respect to e, the introduction of a background risk in this setting stimulates self-protection provided the DT DM is dual prudent.

Next, consider the case $2\varepsilon > \ell$. In this case,

$$\begin{aligned} V[S(e)] &= h(p(e)/2)(w_0 - \ell - \varepsilon - e) + (h(1/2) - h(p(e)/2))(w_0 - \varepsilon - e) \\ &\quad + (h((1+p(e))/2) - h(1/2))(w_0 - \ell + \varepsilon - e) \\ &\quad + (1 - h((1+p(e)/2)))(w_0 + \varepsilon - e), \end{aligned}$$

and the first-order condition is

$$\begin{aligned} \frac{\mathrm{d}V[S(e)]}{\mathrm{d}e} &= (p'(e)/2)h'(p(e)/2)\left((w_0 - \ell - \varepsilon - e) - (w_0 - \varepsilon - e)\right) \\ &\quad + (p'(e)/2)h'((1+p(e))/2)\left((w_0 - \ell + \varepsilon - e) - (w_0 + \varepsilon - e)\right) \\ &\quad - 1 = 0, \end{aligned}$$

which, after obvious simplifications, reduces to

$$-(1/2)p'(e)\ell\left(h'(p(e)/2) + h'((1+p(e))/2)\right) - 1 = 0. \tag{5.5}$$

Recall that in the absence of any background risk, the first-order condition equals (5.3). Hence, noting that the difference between the left-hand side of (5.5) and the left-hand side of (5.3) equals

$$-(1/2)p'(e)\ell\left(h'(p(e)/2) - 2h'(p(e)) + h'((1+p(e))/2)\right), \tag{5.6}$$

and maintaining the assumption that $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) = 1/2$, we find that the impact of the background risk is again linked to the sign of (5.4) which, in turn, depends on the sign of h'''.

Thus, under the maintained assumption that $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) = 1/2$, $h''' \geq 0$ guarantees that the marginal benefit of self-protection increases upon the introduction of an independent background risk. Therefore, the background risk stimulates self-protection under dual prudence.[20]

Now suppose $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) \neq 1/2$. In particular, suppose $2\varepsilon < \ell$, and $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) < 1/2$. The introduction of the background risk is then linked to the sign of

$$\begin{aligned}
&- h'(p(e)/2) + 2h'(p(e)) - h'((1+p(e))/2) \\
&= [h'(p(e)) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(1/2)] - [h'(1/2) - h'(p(e))].
\end{aligned}$$

Under concavity of h', $[h'(p(e)) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(1/2)] \geq 0$. Furthermore, under concavity of h, $-[h'(1/2) - h'(p(e))] \geq 0$. So, if $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) < 1/2$, concavity of h and h' leads (when multiplied by $p'(e)\varepsilon$) to a negative impact in the first-order condition (5.2), hence a reduction of effort, in view of the concavity of V.

Next, with $2\varepsilon < \ell$, suppose $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) > 1/2$. The introduction of the background risk is then linked to the sign of

$$\begin{aligned}
&- h'(p(e)/2) + 2h'(p(e)) - h'((1+p(e))/2) \\
&= [h'(1/2) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(p(e))] - [h'(1/2) - h'(p(e))].
\end{aligned}$$

Under convexity of h', $[h'(1/2) - h'(p(e)/2)] - [h'((1+p(e))/2) - h'(p(e))] \leq 0$. Furthermore, under concavity of h, $-[h'(1/2) - h'(p(e))] \leq 0$. So, if $-p'(e)h'(p(e))\ell - 1 = 0$ when $p(e) > 1/2$, concavity of h and convexity of h' lead to a positive impact in the first-order condition (5.2), hence an increase in effort. The case $2\varepsilon > \ell$ follows similarly, in view of (5.6).

6 Dual Risk Apportionment

In this section, we develop the dual story as illustrated in Section 3 in full generality. Throughout this section we consider n-state lotteries and assume all states to have equal probability of

[20]There is a difference between the two cases. When $2\varepsilon < \ell$, the convexity of h' adds a marginal benefit that depends on the size of ε compared to the situation of no background risk; cf. (5.2). When $2\varepsilon > \ell$, the convexity of h' adds a marginal benefit that is independent of ε but depends on the size of ℓ compared to the situation of no background risk; cf. (5.6).

occurrence $1/n$, $n \in \mathbb{N}$. We permit that the increments in the outcomes when moving to an adjacent state are equal to zero. This can be interpreted to yield lotteries with unequal state probabilities.

6.1 The Third Order: Dual Prudence

We start by providing a simple class of lottery pairs such that the direction of preference between these lottery pairs is equivalent to signing the third derivative of the probability weighting function under DT. Consider

$$\mathcal{G}^{(3)} = [\ \delta, 1/n\ ;\ \ldots;\ -\delta, 1/n\ ] \quad \text{and} \quad \mathcal{B}^{(3)} = [\ -\delta', 1/n\ ;\ \ldots;\ \delta', 1/n\ ],$$

with $\delta, \delta' \geq 0$. The acronyms "$\mathcal{G}$" and "$\mathcal{B}$" refer to "good" and "bad". Upon "attaching" $\mathcal{G}^{(3)}$ and $\mathcal{B}^{(3)}$ state-wise to an arbitrarily given initial lottery that has at least three states such that the state-wise addition is feasible, we generate $D^{(3)}$ if $\mathcal{G}^{(3)}$ state-wise precedes $\mathcal{B}^{(3)}$ and we generate $C^{(3)}$ if $\mathcal{B}^{(3)}$ state-wise precedes $\mathcal{G}^{(3)}$. Attaching the good, $\mathcal{G}^{(3)}$, induces a squeeze, while attaching the bad, $\mathcal{B}^{(3)}$, induces an anti-squeeze.

To emphasize the generality of these transformations, we note that: (i) δ and δ' may differ; (ii) the number of states that "…" denotes in $\mathcal{G}^{(3)}$ and in $\mathcal{B}^{(3)}$ may differ; (iii) the number of states that "…" denotes in $\mathcal{G}^{(3)}$ and in $\mathcal{B}^{(3)}$ may be equal to zero; (iv) the spacing (i.e., the number of states) between $\mathcal{G}^{(3)}$ and $\mathcal{B}^{(3)}$ on the one hand and $\mathcal{B}^{(3)}$ and $\mathcal{G}^{(3)}$ on the other hand when attaching them to the initial lottery may differ; (v) this spacing may be negative, so that the good and the bad partially overlap, but $\mathcal{G}^{(3)}$ strictly precedes $\mathcal{B}^{(3)}$ by at least one state for $D^{(3)}$ and $\mathcal{B}^{(3)}$ strictly precedes $\mathcal{G}^{(3)}$ by at least one state for $C^{(3)}$. All these generalities are permitted as long as the ranking of outcomes remains unaffected by the squeezes and anti-squeezes and the outcomes of the resulting lotteries remain non-negative. These generalities are in part thanks to the fact that we don't compare the changed lotteries to the initial lottery they are generated from. Rather we compare two altered lotteries: one has the good precede the bad, the other the reverse.

We say that an individual is "dual prudent" if, for any such lottery pair $(C^{(3)}, D^{(3)})$, he prefers $D^{(3)}$ to $C^{(3)}$. The following theorem shows that within DT this is guaranteed by a positive sign of h'''. The individual's type of behavior corresponding to these lottery preferences, and their higher-order generalizations that we develop below, may be termed "dual risk apportionment". Much like under primal risk apportionment, the apportionment of harms within a lottery mitigates the detrimental effects for such an individual.

Theorem 6.1 *If $C^{(3)}$ and $D^{(3)}$ are generated by the transformations described above, then $D^{(3)}$ is preferred (dispreferred) to $C^{(3)}$ by any DT DM with $h''' \geq 0$ ($h''' \leq 0$).*

In the remainder of this subsection, we consider a parsimonious subclass of lottery pairs $C^{(3)}$ and $D^{(3)}$ that turns out to be already sufficient for the purpose of signing h'''. Let $n \geq 3$ and consider

$$\begin{aligned} C_n^{(3)} = [\ & x_1, 1/n\ ;\ \ldots;\ x_j, 1/n\ ; \\ & \mathbf{x_{j+1}}, \mathbf{1/n}\ ;\ \mathbf{x_{j+2}}, \mathbf{1/n}\ ;\ \mathbf{x_{j+3}}, \mathbf{1/n}\ ; \\ & x_{j+4}, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

The three states in bold need to be present for any given $n(\geq 3)$, the remaining states are added arbitrarily until the state probabilities sum up to 1. The increments in the outcomes when moving to an adjacent higher state of $C_n^{(3)}$ are allowed to be arbitrarily non-negative and state-dependent, as long as the squeezes and anti-squeezes performed below do not change the ranking of outcomes.[21]

Then, we (directly) construct $D_n^{(3)}$ from $C_n^{(3)}$ by attaching to the three states in bold

$$C2D_n^{(3)} = [\ 1/M, 1/n\ ;\ -2/M, 1/n\ ;\ 1/M, 1/n\ ].$$

[21] This is accomplished by restricting M in the squeezes and anti-squeezes explicated below accordingly.

This yields $D_n^{(3)}$ given by

$$D_n^{(3)} = [\ x_1, 1/n\ ;\ \ldots;\ x_j, 1/n\ ;\ \mathbf{x_{j+1} + 1/M, 1/n}\ ;\ \mathbf{x_{j+2} - 2/M, 1/n}\ ;\ \mathbf{x_{j+3} + 1/M, 1/n}\ ;\ x_{j+4}, 1/n;\ \ldots\ ;\ ., 1/n\ ].$$

Clearly, $C_n^{(3)}$ and $D_n^{(3)}$ occur as a subclass of $C^{(3)}$ and $D^{(3)}$. To see this, consider an initial lottery $L_n^{(3)}$ given by

$$L_n^{(3)} = [\ x_1, 1/n\ ;\ \ldots;\ x_j, 1/n\ ;\ \mathbf{x_{j+1} + 1/(2M), 1/n}\ ;\ \mathbf{x_{j+2} - 1/M, 1/n}\ ;\ \mathbf{x_{j+3} + 1/(2M), 1/n}\ ;\ x_{j+4}, 1/n;\ \ldots\ ;\ ., 1/n\ ],$$

and generate $D_n^{(3)}$ ($C_n^{(3)}$) by attaching the good (bad), $\mathcal{G}^{(3)}$ ($\mathcal{B}^{(3)}$), given by

$$\mathcal{G}^{(3)} = [\ 1/(2M), 1/n\ ;\ -1/(2M), 1/n\ ] \quad \text{and} \quad \mathcal{B}^{(3)} = [\ -1/(2M), 1/n\ ;\ 1/(2M), 1/n\ ],$$

to the first two bold states of $L_n^{(3)}$, and attaching the bad (good), $\mathcal{B}^{(3)}$ ($\mathcal{G}^{(3)}$), to the last two bold states of $L_n^{(3)}$. To keep the required number of states as small as possible and enhance parsimony, sufficient for our purpose of signing h''', we squeeze and anti-squeeze at an overlapping state, just like in Section 3.

The following theorem shows that within DT the preference towards the class of lottery pairs $D_n^{(3)}$ and $C_n^{(3)}$ signs h''':

Theorem 6.2 *If, for any $n \geq 3$, a DT DM prefers (disprefers) $D_n^{(3)}$ to $C_n^{(3)}$, then $h''' \geq 0$ ($h''' \leq 0$).*

6.2 The Fourth Order: Dual Temperance

Next, we provide a simple class of lottery pairs such that the direction of preference between these lottery pairs is equivalent to signing the fourth derivative of the probability weighting function under DT. Consider

$$\mathcal{G}^{(4)} = [\ \delta, 1/n\ ;\ \ldots;\ -\delta, 1/n\ ;\ \ldots;\ -\delta, 1/n\ ;\ \ldots;\ \delta, 1/n\ ] \quad \text{and}$$

$$\mathcal{B}^{(4)} = [\ -\delta', 1/n\ ;\ \ldots;\ \delta', 1/n\ ;\ \ldots;\ \delta', 1/n\ ;\ \ldots;\ -\delta', 1/n\ ],$$

with $\delta, \delta' \geq 0$. Upon attaching $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ to an arbitrarily given initial lottery that has at least four states such that the state-wise addition is feasible, we generate $D^{(4)}$ if $\mathcal{G}^{(4)}$ state-wise precedes $\mathcal{B}^{(4)}$ and we generate $C^{(4)}$ if $\mathcal{B}^{(4)}$ state-wise precedes $\mathcal{G}^{(4)}$. We note that $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ can each be interpreted as simple concatenations of same-sized but exactly opposite $\mathcal{G}^{(3)}$ and $\mathcal{B}^{(3)}$. Thus, we define the fourth order transformations by simple iterations of the third order ones.

To emphasize the generality of these transformations, we note that: (i) δ and δ' may differ; (ii) the number of states that "..." denotes may differ within and between $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$, but both $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ need to be symmetrical in the sense that the first and third "..." within $\mathcal{G}^{(4)}$ denote the same number of states and similarly for $\mathcal{B}^{(4)}$ (which is compatible with $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ each being concatenations of same-sized but exactly opposite $\mathcal{G}^{(3)}$ and $\mathcal{B}^{(3)}$); (iii) the number of states that the first and third "..." denote in $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ is non-negative and may be equal to zero, while the number of states that the middle (second) "..." denotes is larger than or equal to minus one;[22] (iv) the spacing (i.e., the number of states) between $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ on the one hand and $\mathcal{B}^{(4)}$ and $\mathcal{G}^{(4)}$ on the other hand when attaching them to the initial lottery may differ; (v) this spacing may be negative, so that the good and the bad partially overlap, but $\mathcal{G}^{(4)}$ strictly precedes $\mathcal{B}^{(4)}$ by at least one state for $D^{(4)}$ and $\mathcal{B}^{(4)}$ strictly precedes $\mathcal{G}^{(4)}$ by at least one state for $C^{(4)}$. All these generalities are permitted as long as the ranking of outcomes remains unaffected and the outcomes of the resulting lotteries remain non-negative.

We say that an individual is "dual temperate" if, for any such lottery pair $(C^{(4)}, D^{(4)})$, he prefers $D^{(4)}$ to $C^{(4)}$. The following theorem shows that within DT this is guaranteed by a negative sign of h''''.

[22] If the number of states that "..." represents is -1, the middle two adjacent states overlap, thus attaching -2δ for $\mathcal{G}^{(4)}$ and $2\delta'$ for $\mathcal{B}^{(4)}$ to a state with probability $1/n$.

Theorem 6.3 *If $C^{(4)}$ and $D^{(4)}$ are generated by the transformations described above, then $D^{(4)}$ is preferred (dispreferred) to $C^{(4)}$ by any DT DM with $h'''' \leq 0$ ($h'''' \geq 0$).*

As in Section 6.1, let us in the remainder of this subsection consider a parsimonious subclass of lottery pairs $C^{(4)}$ and $D^{(4)}$, already sufficient for our purpose of signing h''''. Let $n \geq 4$ and consider

$$C_n^{(4)} = [\ x_1, 1/n\ ;\ \ldots;\ x_j, 1/n\ ;\ \mathbf{x_{j+1}, 1/n}\ ;\ \mathbf{x_{j+2}, 1/n}\ ;\ \mathbf{x_{j+3}, 1/n}\ ;\ \mathbf{x_{j+4}, 1/n}\ ;\ x_{j+5}, 1/n;\ \ldots\ ;\ ., 1/n\ ].$$

The four states in bold need to be present for any given $n(\geq 4)$, the remaining states are added arbitrarily until the state probabilities sum up to 1. The increments in the outcomes when moving to an adjacent higher state of $C_n^{(4)}$ are again allowed to be arbitrarily non-negative and state-dependent, as long as the transformations conducted below do not change the ranking of outcomes.[23]

Then, we (directly) construct $D_n^{(4)}$ from $C_n^{(4)}$ by attaching to the four states in bold

$$C2D_n^{(4)} = [\ 1/M, 1/n\ ;\ -3/M, 1/n\ ;\ 3/M, 1/n\ ;\ -1/M, 1/n\ ].$$

This yields $D_n^{(4)}$ given by

$$D_n^{(4)} = [\ x_1, 1/n\ ;\ \ldots;\ x_j, 1/n\ ;\ \mathbf{x_{j+1} + 1/M, 1/n}\ ;\ \mathbf{x_{j+2} - 3/M, 1/n}\ ;\ \mathbf{x_{j+3} + 3/M, 1/n}\ ;\ \mathbf{x_{j+4} - 1/M, 1/n}\ ;\ x_{j+5}, 1/n;\ \ldots\ ;\ ., 1/n\ ].$$

Clearly, $C_n^{(4)}$ and $D_n^{(4)}$ occur as a subclass of $C^{(4)}$ and $D^{(4)}$. To see this, consider an initial

[23] This is accomplished by restricting M in the transformations explicated below accordingly.

lottery $L_n^{(4)}$ given by

$$L_n^{(4)} = [\ x_1, 1/n\ ;\ \ldots;\ x_j, 1/n\ ;\ \mathbf{x_{j+1} + 1/(2M), 1/n}\ ;\ \mathbf{x_{j+2} - 3/(2M), 1/n}\ ;\ \mathbf{x_{j+3} + 3/(2M), 1/n}\ ;\ \mathbf{x_{j+4} - 1/(2M), 1/n}\ ;\ x_{j+5}, 1/n;\ \ldots\ ;\ ., 1/n\ ],$$

and generate $D_n^{(4)}$ ($C_n^{(4)}$) by attaching the good (bad), $\mathcal{G}^{(4)}$ ($\mathcal{B}^{(4)}$), given by

$$\mathcal{G}^{(4)} = [\ 1/(2M), 1/n\ ;\ -1/M, 1/n\ ;\ 1/(2M), 1/n\ ] \quad \text{and}$$
$$\mathcal{B}^{(4)} = [\ -1/(2M), 1/n\ ;\ 1/M, 1/n\ ;\ -1/(2M), 1/n\ ],$$

to the first three bold states of $L_n^{(4)}$, and attaching the bad (good), $\mathcal{B}^{(4)}$ ($\mathcal{G}^{(4)}$), to the last three bold states of $L_n^{(4)}$. To keep the required number of states as small as possible and enhance parsimony, sufficient for our purpose of signing h'''', we again conduct transformations at overlapping states, just like in Section 6.1 and Section 3.

The following theorem shows that within DT the preference towards the class of lottery pairs $D_n^{(4)}$ and $C_n^{(4)}$ signs h'''':

Theorem 6.4 *If, for any $n \geq 4$, a DT DM prefers (disprefers) $D_n^{(4)}$ to $C_n^{(4)}$, then $h'''' \leq 0$ ($h'''' \geq 0$).*

6.3 The m^{th} Order

In this section, we systematically construct simple nested classes of lottery pairs such that the direction of preference between these lottery pairs is equivalent to signing the m^{th} derivative of the probability weighting function under DT. We start at the second order and proceed to construct all higher orders of dual risk apportionment by simple iteration, as follows. Consider

$$\mathcal{G}^{(2)} = [\ \delta, 1/n\ ] \quad \text{and} \quad \mathcal{B}^{(2)} = [\ -\delta', 1/n\ ],$$

with $\delta, \delta' \geq 0$. Taking the $(m-1)^{\text{th}}$ order as a starting point, we construct the class of lottery pairs for the m^{th} order in two steps. Let $n \geq m$. Then:

(i) Concatenate a same-sized but exactly opposite (in sign) $\mathcal{G}^{(m-1)}$ and $\mathcal{B}^{(m-1)}$, with $\mathcal{G}^{(m-1)}$ state-wise preceding $\mathcal{B}^{(m-1)}$ by at least one state. This generates $\mathcal{G}^{(m)}$. Similarly, concatenate another same-sized but exactly opposite $\mathcal{G}^{(m-1)}$ and $\mathcal{B}^{(m-1)}$, with $\mathcal{B}^{(m-1)}$ state-wise preceding $\mathcal{G}^{(m-1)}$ by at least one state. This generates $\mathcal{B}^{(m)}$.

(ii) Attach $\mathcal{G}^{(m)}$ and $\mathcal{B}^{(m)}$ to an arbitrarily given n-state initial lottery. We generate $D^{(m)}$ if $\mathcal{G}^{(m)}$ state-wise precedes $\mathcal{B}^{(m)}$ and we generate $C^{(m)}$ if $\mathcal{B}^{(m)}$ state-wise precedes $\mathcal{G}^{(m)}$.

In the transformations to $D^{(m)}$ and $C^{(m)}$ we require that the ranking of outcomes remains unaffected and the outcomes of the resulting lotteries remain non-negative.

Then, we state the following theorem:

Theorem 6.5 *Let $m \geq 2$. If $C^{(m)}$ and $D^{(m)}$ are generated by the transformations described above, then $D^{(m)}$ is preferred (dispreferred) to $C^{(m)}$ by any DT DM with $(-1)^{m-1}h^{(m)} \geq 0$ ($(-1)^{m-1}h^{(m)} \leq 0$).*

Just like in Sections 6.1 and 6.2 we consider henceforth parsimonious subclasses of lottery pairs $C^{(m)}$ and $D^{(m)}$ that are already sufficient for signing $h^{(m)}$. We start at the second order and proceed by simple iteration. Let $n \geq m \geq 2$ and consider

$$\begin{aligned} C_n^{(m)} =[\ & x_1, 1/n\ ;\ \ldots;\ x_j, 1/n\ ; \\ & \mathbf{x_{j+1}}, \mathbf{1/n}\ ;\ \ldots;\ \mathbf{x_{j+m}}, \mathbf{1/n}\ ; \\ & x_{j+m+1}, 1/n;\ \ldots\ ;\ ., 1/n\ ]. \end{aligned}$$

The m states in bold need to be present for any $n(\geq m)$, the remaining states are added arbitrarily until the state probabilities sum up to 1. The increments in the outcomes when moving to an adjacent higher state of $C_n^{(m)}$ are allowed to be arbitrarily non-negative and state-dependent, as long as the transformations conducted below do not change the ranking of outcomes. Taking the $(m-1)^{\text{th}}$ order as a starting point, we construct the class of lottery pairs for the m^{th} order in two steps:

(i) Concatenate a same-sized but exactly opposite (in sign) $\mathcal{G}^{(m-1)}$ and $\mathcal{B}^{(m-1)}$, with $\mathcal{G}^{(m-1)}$ state-wise preceding $\mathcal{B}^{(m-1)}$ by exactly one state. This generates $\mathcal{G}^{(m)}$. Next, concatenate the same $\mathcal{G}^{(m-1)}$ and $\mathcal{B}^{(m-1)}$ with $\mathcal{B}^{(m-1)}$ state-wise preceding $\mathcal{G}^{(m-1)}$ by exactly one state. This generates $\mathcal{B}^{(m)}$. (For $m = 2$, we set $\mathcal{G}^{(2)} = [\ \delta, 1/n\ ]$ and $\mathcal{B}^{(2)} = [\ -\delta, 1/n\ ]$, $\delta > 0$.)

(ii) Generate $D_n^{(m)}$ by attaching $\mathcal{G}^{(m)}$ to the first $(m-1)$ bold states of $C_n^{(m)}$ and attaching $\mathcal{B}^{(m)}$ to the last $(m-1)$ bold states of $C_n^{(m)}$.

In the transformations from $C_n^{(m)}$ to $D_n^{(m)}$ we require that the ranking of outcomes remains unaffected and the outcomes of the resulting lotteries remain non-negative. For $m \geq 3$, we perform our sequences of squeezes and anti-squeezes at overlapping states to keep the required number of states minimal. This yields a parsimonious approach already sufficient for signing $h^{(m)}$.

Applying this construction of concatenation, one easily verifies that, up to the sixth order,

$$\mathcal{G}^{(3)} = [\ \delta, 1/n\ ;\ -\delta, 1/n\ ]$$
$$\mathcal{G}^{(4)} = [\ \delta, 1/n\ ;\ -2\delta, 1/n\ ;\ \delta, 1/n\ ]$$
$$\mathcal{G}^{(5)} = [\ \delta, 1/n\ ;\ -3\delta, 1/n\ ;\ 3\delta, 1/n\ ;\ -\delta, 1/n\ ]$$
$$\mathcal{G}^{(6)} = [\ \delta, 1/n\ ;\ -4\delta, 1/n\ ;\ 6\delta, 1/n\ ;\ -4\delta, 1/n\ ;\ \delta, 1/n\ ].$$

Finally, the following theorem shows that within DT the preference towards the class of lottery pairs $D_n^{(m)}$ and $C_n^{(m)}$ signs $h^{(m)}$:

Theorem 6.6 *Let $m \geq 2$. If, for any $n \geq m$, a DT DM prefers (disprefers) $D_n^{(m)}$ to $C_n^{(m)}$, then $(-1)^{m-1}h^{(m)} \geq 0$ ($(-1)^{m-1}h^{(m)} \leq 0$).*

7 Conclusion

Starting with Menezes, Geiss and Tressler (1980), many papers have been devoted to an interpretation of the signs of the successive derivatives of the utility function within the EU model.

In this paper we have developed a model free story of preferences towards particular nested classes of lottery pairs that is appropriate to satisfy the specific requirements of the DT model. The story yields an intuitive interpretation, and full characterization, of the dual counterparts of such concepts as prudence and temperance. The direction of preference between the nested lottery pairs that are provided by the story is equivalent to signing the m^{th} derivative of the probability weighting function within DT.

We have analyzed implications of our results for portfolio choice, which appear to stand in sharp contrast to familiar implications under the EU model. We have also shown that, where the sign of the third derivative of the utility function is connected to a savings problem, the sign of the third derivative of the probability weighting function may be naturally linked to a self-protection problem.

Because the primal and dual stories have several aspects in common, some of the implications of the primal story can potentially be extended to a dual world. For instance, because it is also simple, the dual story should be as amenable to experimentation as the primal story. Another promising route for future research is that the dual story can serve as a building block on the basis of which it should be possible to obtain related interpretations for more general models of choice under risk (and ambiguity). Indeed, now that the dual story has been told, future research can develop an interpretation to the signs of the successive derivatives of both the utility function and the probability weighting function in the rank-dependent utility (RDU) model of Quiggin (1982) and in prospect theory of Tversky and Kahneman (1992). For example, one may verify that the direction of preference between the *intersection* of lottery pairs generated by primal and dual risk apportionment signs the successive derivatives of the utility and probability weighting functions *jointly* under RDU. This would provide a test for the null hypothesis that both $U^{(3)} \geq 0$ and $h^{(3)} \geq 0$ in the RDU environment, as is often assumed in parametric specifications of the utility and probability weighting functions. Furthermore, subsets of lottery pairs generated by dual risk apportionment that share the same final wealth outcomes can be used to test the null hypothesis that (only) $h^{(3)} \geq 0$ in the RDU setting.

As such, this paper represents a first step, and paves new ways, towards the development of higher order risk attitudes in non-EU theories, as explicitly desired by Deck and Schlesinger (2010).

Appendix: Proofs

Proof of Proposition 2.1. Let $h(p) = \alpha p - \beta p^2$. Note that $h(0) = 0$. To have $h(1) = 1$, $\alpha - \beta = 1$ should hold. Hence, $h(p) = (1 + \beta)p - \beta p^2$, so that $h'(p) = 1 + \beta - 2\beta p$. To have $h'(1) \geq 0$ (hence $h'(p) \geq 0$ whenever $h''(p) \leq 0$), $\beta \leq 1$ should hold. Furthermore, $h''(p) = -2\beta$ so $h'' \leq 0$ if $\beta \geq 0$. In sum, $0 \leq \beta \leq 1$ and $\alpha = 1 + \beta$. With such a probability weighting function,

$$V[A] = ((1/4)(1 + \beta) - (1/16)\beta)0 + (1 - (1/4)(1 + \beta) + (1/16)\beta)2 = 3/2 - (3/8)\beta.$$

$$V[B] = ((3/4)(1 + \beta) - (9/16)\beta)1 + (1 - (3/4)(1 + \beta) + (9/16)\beta)3 = 3/2 - (3/8)\beta.$$

□

Proof of Proposition 2.3. With a probability weighting function that satisfies the same properties as in the proof of Proposition 2.1,

$$\begin{aligned} V[\tilde{A}] &= ((1/4)(1 + \beta) - (1/16)\beta)2 \\ &\quad + (((1/2)(1 + \beta) - (1/4)\beta) - ((1/4)(1 + \beta) - (1/16)\beta))6 \\ &\quad + (1 - ((1/2)(1 + \beta) - (1/4)\beta))10 = 7 - (7/4)\beta. \\ V[\tilde{B}] &= ((1/2)(1 + \beta) - (1/4)\beta)4 \\ &\quad + (((3/4)(1 + \beta) - (9/16)\beta) - ((1/2)(1 + \beta) - (1/4)\beta))8 \\ &\quad + (1 - ((3/4)(1 + \beta) - (9/16)\beta))12 = 7 - (7/4)\beta. \end{aligned}$$

□

Proof of Theorem 6.1. We first note that the probabilities of generating the minimal outcome in a two-shot experiment (i.e., in two independent draws) for the states of an arbitrarily given n-state lottery A with equal state probabilities are

$$(2n - 1)/n^2, \ldots, 7/n^2, 5/n^2, 3/n^2, 1/n^2,$$

respectively. Thus, the second dual moment of A is given by

$$\mathrm{E}\left[\min(A_1, A_2)\right] = (1/n^2)x_n + (3/n^2)x_{n-1} + (5/n^2)x_{n-2} + (7/n^2)x_{n-3} + \cdots + ((2n-1)/n^2)x_1.$$

Hence, one readily verifies that

$$\begin{aligned} \mathrm{E}\left[C^{(3)}\right] &= \mathrm{E}\left[D^{(3)}\right], \\ \mathrm{E}\left[\min(C_1^{(3)}, C_2^{(3)})\right] &= \mathrm{E}\left[\min(D_1^{(3)}, D_2^{(3)})\right]. \end{aligned}$$

Indeed, attaching $\mathcal{G}^{(3)}$ and $\mathcal{B}^{(3)}$ with $\mathcal{G}^{(3)}$ state-wise preceding $\mathcal{B}^{(3)}$ versus attaching $\mathcal{B}^{(3)}$ and $\mathcal{G}^{(3)}$ with $\mathcal{B}^{(3)}$ state-wise preceding $\mathcal{G}^{(3)}$ has the same (incremental) impact on the dual moments up to the second order. Furthermore, $F_{C^{(3)}}$ surpasses $F_{D^{(3)}}$ before crossing twice, so that $D^{(3)}$ dominates $C^{(3)}$ in third-degree dual stochastic dominance; see e.g., Proposition 4.9 of Wang and Young (1998).

It then follows from a Taylor expansion of the "dual utility premium",

$$V\left[D^{(3)}\right] - V\left[C^{(3)}\right],$$

(see Theorem 4.4 of Wang and Young (1998)) that when the dual moments are equal up to the second order, $h''' \geq 0$ ($h''' \leq 0$) implies that the dual utility premium above is non-negative (non-positive). This proves the stated result. □

Proof of Theorem 6.2. Since h'' is differentiable on $(0, 1)$, the third derivative of h being non-negative is equivalent to requiring that

$$\begin{aligned} &([h(p_+) - h(p_0)] - [h(p_0) - h(p_-)]) - ([h(p_0) - h(p_-)] - [h(p_-) - h(p_{--})]) \\ &\qquad = h(p_+) - 3h(p_0) + 3h(p_-) - h(p_{--}) \geq 0, \end{aligned}$$

for any four equidistant probabilities $0 \leq p_{--} \leq p_- \leq p_0 \leq p_+ \leq 1$. We will show that this condition is satisfied whenever $D_n^{(3)}$ is preferred to $C_n^{(3)}$ for any $n \geq 3$ within DT. (The implication $h''' \leq 0$ follows similarly.)

We note that the DT evaluation of a lottery A with n outcomes $x_0 = 0 \leq x_1 \leq \cdots \leq x_n$ is given by

$$\begin{aligned} V[A] &= \int_0^\infty x \,\mathrm{d}\left(1 - \bar{h}(1 - F_A(x))\right) = -\sum_{i=1}^n x_i \left(\bar{h}(1 - F_A(x_i)) - \bar{h}(1 - F_A(x_{i-1}))\right) \\ &= \int_0^\infty \bar{h}(1 - F_A(x))\,\mathrm{d}x = \sum_{i=1}^n \bar{h}(1 - F_A(x_{i-1}))(x_i - x_{i-1}), \end{aligned}$$

with $\bar{h}(p) = 1 - h(1-p)$. Observe that $\bar{h} : [0,1] \to [0,1]$ with $\bar{h}(0) = 0$, $\bar{h}(1) = 1$, $\bar{h}' \geq 0$; and that $\bar{h}'' \geq 0$ is equivalent to $h'' \leq 0$. Here, $F_A(x_0) = 0$ by convention, so $1 - F_A(x_0) = \bar{h}(1 - F_A(x_0)) = 1$. Hence,

$$\begin{aligned} V[C_n^{(3)}] =& \sum_{i=1}^{j} \bar{h}(1 - F_{C_n^{(3)}}(x_{i-1}))(x_i - x_{i-1}) + \sum_{i=j+1}^{j+4} \bar{h}(1 - F_{C_n^{(3)}}(x_{i-1}))(x_i - x_{i-1}) \\ &+ \sum_{i=j+5}^{n} \bar{h}(1 - F_{C_n^{(3)}}(x_{i-1}))(x_i - x_{i-1}) \\ =:& \Sigma_1^{C_n^{(3)}} + \Sigma_2^{C_n^{(3)}} + \Sigma_3^{C_n^{(3)}}, \end{aligned}$$

with j corresponding to the state preceding the three states in bold that are always present in $C_n^{(3)}$, and similarly for $D_n^{(3)}$. Notice that

$$\Sigma_1^{C_n^{(3)}} = \Sigma_1^{D_n^{(3)}}, \qquad \Sigma_3^{C_n^{(3)}} = \Sigma_3^{D_n^{(3)}}.$$

Furthermore,

$$\begin{aligned} \Sigma_2^{C_n^{(3)}} =& (x_{j+1} - x_j)\bar{h}(1 - F(x_j)) + (x_{j+2} - x_{j+1})\bar{h}(1 - F(x_{j+1})) \\ &+ (x_{j+3} - x_{j+2})\bar{h}(1 - F(x_{j+2})) + (x_{j+4} - x_{j+3})\bar{h}(1 - F(x_{j+3})), \end{aligned}$$

and

$$\begin{aligned} \Sigma_2^{D_n^{(3)}} =& (x_{j+1} - x_j + 1/M)\bar{h}(1 - F(x_j)) + (x_{j+2} - x_{j+1} - 3/M)\bar{h}(1 - F(x_{j+1})) \\ &+ (x_{j+3} - x_{j+2} + 3/M)\bar{h}(1 - F(x_{j+2})) + (x_{j+4} - x_{j+3} - 1/M)\bar{h}(1 - F(x_{j+3})), \end{aligned}$$

suppressing the indices $C_n^{(3)}$ and $D_n^{(3)}$ for convenience. Define, for given $n \geq 3$, the probability q, $0 \leq q \leq 1 - 3/n$, such that

$$\begin{aligned} F(x_j) = q, \qquad F(x_{j+1}) &= q + 1/n, \qquad F(x_{j+2}) = q + 2/n, \\ F(x_{j+3}) &= q + 3/n, \qquad F(x_{j+4}) = q + 4/n. \end{aligned}$$

Then,

$$\begin{aligned} \Sigma_2^{C_n^{(3)}} &= (x_{j+1} - x_j)\bar{h}(1-q) + (x_{j+2} - x_{j+1})\bar{h}(1-q-1/n) \\ &\quad + (x_{j+3} - x_{j+2})\bar{h}(1-q-2/n) + (x_{j+4} - x_{j+3})\bar{h}(1-q-3/n), \end{aligned}$$

and

$$\begin{aligned} \Sigma_2^{D_n^{(3)}} &= (x_{j+1} - x_j + 1/M)\bar{h}(1-q) + (x_{j+2} - x_{j+1} - 3/M)\bar{h}(1-q-1/n) \\ &\quad + (x_{j+3} - x_{j+2} + 3/M)\bar{h}(1-q-2/n) + (x_{j+4} - x_{j+3} - 1/M)\bar{h}(1-q-3/n). \end{aligned}$$

Upon defining p_{--}, p_-, p_0, p_+ such that

$$\begin{aligned} 3/n \leq 1 - q = p_+ \leq 1, \qquad 1 - q - 1/n = p_0, \\ 1 - q - 2/n = p_-, \qquad 1 - q - 3/n = p_{--}, \end{aligned}$$

it then follows from the arbitrariness of $n \geq 3$, hence the arbitrariness of $0 \leq q \leq 1 - 3/n$, that

$$\Sigma_2^{D_n^{(3)}} - \Sigma_2^{C_n^{(3)}} = (1/M)\bar{h}(p_+) - (3/M)\bar{h}(p_0) + (3/M)\bar{h}(p_-) - (1/M)\bar{h}(p_{--}) \geq 0,$$

for any four equidistant probabilities $0 \leq p_{--} \leq p_- \leq p_0 \leq p_+ \leq 1$, whenever $D_n^{(3)}$ is preferred to $C_n^{(3)}$ for any $n \geq 3$. We note that $\Sigma_2^{D_n^{(3)}} - \Sigma_2^{C_n^{(3)}}$ is the DT analog of the utility premium in the EU model. Finally, observe that $\bar{h}''' \geq 0$ is equivalent to $h''' \geq 0$. This proves the stated result. □

Proof of Theorem 6.3. By analogy to the proof of Theorem 6.1, we first note that the probabilities of generating the minimal outcome in a three-shot experiment (that is, in three independent draws) for the states of an arbitrarily given n-state lottery A with equal state probabilities are

$$(3(n-1)n+1)/n^3, \ldots, 37/n^3, 19/n^3, 7/n^3, 1/n^3,$$

respectively. Thus, the third dual moment of A is given by

$$\begin{aligned} \mathrm{E}\left[\min(A_1, A_2, A_3)\right] = (1/n^3)x_n + (7/n^3)x_{n-1} + (19/n^3)x_{n-2} + (37/n^3)x_{n-3} \\ + \cdots + ((3(n-1)n+1)/n^3)x_1. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} \mathrm{E}\left[C^{(4)}\right] &= \mathrm{E}\left[D^{(4)}\right], \\ \mathrm{E}\left[\min(C_1^{(4)}, C_2^{(4)})\right] &= \mathrm{E}\left[\min(D_1^{(4)}, D_2^{(4)})\right], \\ \mathrm{E}\left[\min(C_1^{(4)}, C_2^{(4)}, C_3^{(4)})\right] &= \mathrm{E}\left[\min(D_1^{(4)}, D_2^{(4)}, D_3^{(4)})\right]. \end{aligned}$$

Indeed, attaching $\mathcal{G}^{(4)}$ and $\mathcal{B}^{(4)}$ with $\mathcal{G}^{(4)}$ state-wise preceding $\mathcal{B}^{(4)}$ versus attaching $\mathcal{B}^{(4)}$ and $\mathcal{G}^{(4)}$ with $\mathcal{B}^{(4)}$ state-wise preceding $\mathcal{G}^{(4)}$ has the same (incremental) impact on the dual moments up to the third order. Furthermore, $F_{C^{(4)}}$ surpasses $F_{D^{(4)}}$ before crossing three times, so that $D^{(4)}$ dominates $C^{(4)}$ in fourth-degree dual stochastic dominance; see e.g., Proposition 4.9 of Wang and Young (1998).

It then follows from a Taylor expansion of the "dual utility premium",

$$V\left[D^{(4)}\right] - V\left[C^{(4)}\right],$$

(see Theorem 4.4 of Wang and Young (1998)) that when the dual moments are equal up to the third order, $h'''' \leq 0$ ($h'''' \geq 0$) implies that the dual utility premium above is non-negative (non-positive). This proves the stated result. □

Proof of Theorem 6.4. Since h''' is differentiable on $(0, 1)$, the fourth derivative of h being non-positive is equivalent to requiring that

$$
\begin{aligned}
&[([h(p_{++}) - h(p_+)] - [h(p_+) - h(p_0)]) - ([h(p_+) - h(p_0)] - [h(p_0) - h(p_-)])] \\
&\quad - [([h(p_+) - h(p_0)] - [h(p_0) - h(p_-)]) - ([h(p_0) - h(p_-)] - [h(p_-) - h(p_{--})])] \\
&\qquad = h(p_{++}) - 4h(p_+) + 6h(p_0) - 4h(p_-) + h(p_{--}) \leq 0,
\end{aligned}
$$

for any five equidistant probabilities $0 \leq p_{--} \leq p_- \leq p_0 \leq p_+ \leq p_{++} \leq 1$. We will show that this condition is satisfied whenever $D_n^{(4)}$ is preferred to $C_n^{(4)}$ for any $n \geq 4$ within DT. (The implication $h'''' \geq 0$ follows similarly.)

Recall that the DT evaluation of a lottery A with n outcomes $x_0 = 0 \leq x_1 \leq \cdots \leq x_n$ is given by

$$V[A] = \sum_{i=1}^{n} \bar{h}(1 - F_A(x_{i-1}))(x_i - x_{i-1}),$$

with $\bar{h}(p) = 1 - h(1-p)$. Here, $F_A(x_0) = 0$ by convention, so $1 - F_A(x_0) = \bar{h}(1 - F_A(x_0)) = 1$. Hence,

$$
\begin{aligned}
V[C_n^{(4)}] = &\sum_{i=1}^{j} \bar{h}(1 - F_{C_n^{(4)}}(x_{i-1}))(x_i - x_{i-1}) + \sum_{i=j+1}^{j+5} \bar{h}(1 - F_{C_n^{(4)}}(x_{i-1}))(x_i - x_{i-1}) \\
&+ \sum_{i=j+6}^{n} \bar{h}(1 - F_{C_n^{(4)}}(x_{i-1}))(x_i - x_{i-1}) \\
=: &\ \Sigma_1^{C_n^{(4)}} + \Sigma_2^{C_n^{(4)}} + \Sigma_3^{C_n^{(4)}},
\end{aligned}
$$

with j corresponding to the state preceding the four states in bold that are always present in $C_n^{(4)}$, and similarly for $D_n^{(4)}$. Notice that

$$\Sigma_1^{C_n^{(4)}} = \Sigma_1^{D_n^{(4)}}, \qquad \Sigma_3^{C_n^{(4)}} = \Sigma_3^{D_n^{(4)}}.$$

Furthermore,

$$\begin{aligned}\Sigma_2^{C_n^{(4)}} &= (x_{j+1}-x_j)\bar{h}(1-F(x_j)) + (x_{j+2}-x_{j+1})\bar{h}(1-F(x_{j+1})) \\ &\quad + (x_{j+3}-x_{j+2})\bar{h}(1-F(x_{j+2})) + (x_{j+4}-x_{j+3})\bar{h}(1-F(x_{j+3})) \\ &\quad + (x_{j+5}-x_{j+4})\bar{h}(1-F(x_{j+4})),\end{aligned}$$

and

$$\begin{aligned}\Sigma_2^{D_n^{(4)}} &= (x_{j+1}-x_j+1/M)\bar{h}(1-F(x_j)) + (x_{j+2}-x_{j+1}-4/M)\bar{h}(1-F(x_{j+1})) \\ &\quad + (x_{j+3}-x_{j+2}+6/M)\bar{h}(1-F(x_{j+2})) + (x_{j+4}-x_{j+3}-4/M)\bar{h}(1-F(x_{j+3})) \\ &\quad + (x_{j+5}-x_{j+4}+1/M)\bar{h}(1-F(x_{j+4})),\end{aligned}$$

suppressing the indices $C_n^{(4)}$ and $D_n^{(4)}$ for convenience. Define, for given $n \geq 4$, the probability q, $0 \leq q \leq 1-4/n$, such that

$$\begin{aligned} F(x_j) &= q, & F(x_{j+1}) &= q+1/n, & F(x_{j+2}) &= q+2/n, \\ F(x_{j+3}) &= q+3/n, & F(x_{j+4}) &= q+4/n, & F(x_{j+5}) &= q+5/n. \end{aligned}$$

Then,

$$\begin{aligned}\Sigma_2^{C_n^{(4)}} &= (x_{j+1}-x_j)\bar{h}(1-q) + (x_{j+2}-x_{j+1})\bar{h}(1-q-1/n) \\ &\quad + (x_{j+3}-x_{j+2})\bar{h}(1-q-2/n) + (x_{j+4}-x_{j+3})\bar{h}(1-q-3/n) \\ &\quad + (x_{j+5}-x_{j+4})\bar{h}(1-q-4/n), \\ \Sigma_2^{D_n^{(4)}} &= (x_{j+1}-x_j+1/M)\bar{h}(1-q) + (x_{j+2}-x_{j+1}-4/M)\bar{h}(1-q-1/n) \\ &\quad + (x_{j+3}-x_{j+2}+6/M)\bar{h}(1-q-2/n) + (x_{j+4}-x_{j+3}-4/M)\bar{h}(1-q-3/n) \\ &\quad + (x_{j+5}-x_{j+4}+1/M)\bar{h}(1-q-4/n).\end{aligned}$$

Upon defining $p_{--}, p_-, p_0, p_+, p_{++}$ such that

$$4/n \leq 1-q = p_{++} \leq 1, \qquad 1-q-1/n = p_+, \qquad 1-q-2/n = p_0,$$
$$1-q-3/n = p_-, \qquad 1-q-4/n = p_{--},$$

it then follows from the arbitrariness of $n \geq 4$, hence the arbitrariness of $0 \leq q \leq 1-4/n$, that

$$\begin{aligned}\Sigma_2^{D_n^{(4)}} - \Sigma_2^{C_n^{(4)}} &= (1/M)\bar{h}(p_{++}) - (4/M)\bar{h}(p_+) + (6/M)\bar{h}(p_0) - (4/M)\bar{h}(p_-) + (1/M)\bar{h}(p_{--}) \\ &\geq 0,\end{aligned}$$

or equivalently,

$$h(p_{++}) - 4h(p_+) + 6h(p_0) - 4h(p_-) + h(p_{--}) \leq 0,$$

for any five equidistant points $0 \leq p_{--} \leq p_- \leq p_0 \leq p_+ \leq p_{++} \leq 1$, whenever $D_n^{(4)}$ is preferred to $C_n^{(4)}$ for any $n \geq 4$. We note that $\Sigma_2^{D_n^{(4)}} - \Sigma_2^{C_n^{(4)}}$ is again the DT analog of the utility premium in the EU model. This proves the stated result. □

Proof of Theorem 6.5. One can verify that, by construction, the first $m-1$ dual moments satisfy

$$\begin{aligned}\mathrm{E}\left[C^{(m)}\right] &= \mathrm{E}\left[D^{(m)}\right], \\ \mathrm{E}\left[\min(C_1^{(m)}, C_2^{(m)})\right] &= \mathrm{E}\left[\min(D_1^{(m)}, D_2^{(m)})\right], \\ \mathrm{E}\left[\min(C_1^{(m)}, C_2^{(m)}, C_3^{(m)})\right] &= \mathrm{E}\left[\min(D_1^{(m)}, D_2^{(m)}, D_3^{(m)})\right], \\ &\vdots \\ \mathrm{E}\left[\min(C_1^{(m)}, C_2^{(m)}, C_3^{(m)}, \ldots, C_{m-1}^{(m)})\right] &= \mathrm{E}\left[\min(D_1^{(m)}, D_2^{(m)}, D_3^{(m)}, \ldots, D_{m-1}^{(m)})\right],\end{aligned}$$

and that $F_{C^{(m)}}$ surpasses $F_{D^{(m)}}$ before crossing $m-1$ times, so that $D^{(m)}$ dominates $C^{(m)}$ in m^{th}-degree dual stochastic dominance; see e.g., Proposition 4.9 of Wang and Young (1998).

It then follows from a Taylor expansion of the "dual utility premium",

$$V\left[D^{(m)}\right] - V\left[C^{(m)}\right],$$

(see Theorem 4.4 of Wang and Young (1998)) that under the dual moment equalities up to the $(m-1)^{\text{th}}$ order, $(-1)^{m-1}h^{(m)} \geq 0$ ($(-1)^{m-1}h^{(m)} \leq 0$) implies that the dual utility premium above is non-negative (non-positive). This proves the stated result. □

Proof of Theorem 6.6. The proof follows from considering the "dual utility premium" given by

$$\Sigma_2^{D_n^{(m)}} - \Sigma_2^{C_n^{(m)}},$$

defined similarly as their analogs in the proofs of Theorems 6.2 and 6.4, and the definition of $(-1)^{m-1}h^{(m)} \geq 0$. □

References

[1] Baillon, A. (2017). Prudence (and more) with respect to uncertainty and ambiguity. *Economic Journal*, forthcoming.

[2] Chateauneuf, A., T. Gajdos and P.-H. Wilthien (2002). The principle of strong diminishing transfer. *Journal of Economic Theory* 103, 311-333.

[3] Chew, S.H., E. Karni and Z. Safra (1987). Risk aversion in the theory of expected utility with rank dependent probabilities. *Journal of Economic Theory* 42, 370-381.

[4] Chiu, H. (2005). Skewness preference, risk aversion, and the precedence relations on stochastic changes. *Management Science* 51, 1816-1828.

[5] David, H.A. (1981). *Order Statistics*. 2nd Ed., Wiley, New York.

[6] De La Cal, J. and J. Cárcamo (2010). Inverse stochastic dominance, majorization and mean order statistics. *Journal of Applied Probability* 47, 277-292.

[7] Deck, C. and H. Schlesinger (2010). Exploring higher order risk effects. *Review of Economic Studies* 77, 1403-1420.

[8] Deck, C. and H. Schlesinger (2014). Consistency of higher order risk preferences. *Econometrica* 82, 1913-1943.

[9] Dittmar, R.F. (2002). Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns. *Journal of Finance* 57, 369-403.

[10] Ebert, S. and D. Wiesen (2011). Testing for prudence and skewness seeking. *Management Science* 57, 1334-1349.

[11] Ebert, S. and D. Wiesen (2014). Joint measurement of risk aversion, prudence, and temperance. *Journal of Risk and Uncertainty* 48, 231-252.

[12] Eeckhoudt, L.R. and C. Gollier (1995). *Risk: evaluation, management and sharing.* Harvester Wheatsheaf, Hertfordshire.

[13] Eeckhoudt, L.R. and C. Gollier (2005). The impact of prudence on optimal prevention. *Economic Theory* 26, 989-994.

[14] Eeckhoudt, L.R. and H. Schlesinger (2006). Putting risk in its proper place. *American Economic Review* 96, 280-289.

[15] Eeckhoudt L.R., H. Schlesinger and I. Tsetlin (2009). Apportioning of risks via stochastic dominance. *Journal of Economic Theory* 144, 994-1003.

[16] Ehrlich, I. and G. Becker (1972). Market insurance, self insurance and self protection. *Journal of Political Economy* 80, 623-648.

[17] Ekern, S. (1980). Increasing nth degree risk. *Economics Letters* 6, 329-333.

[18] Gollier, C. (1995). The comparative statics of changes in risk revisited. *Journal of Economic Theory* 66, 522-536.

[19] Kahneman, D. and A. Tversky (1979). Prospect theory: An analysis of decision under risk. *Econometrica* 47, 263-292.

[20] Kimball, M.S. (1990). Precautionary saving in the small and in the large. *Econometrica* 58, 53-73.

[21] Mao, J.C.T. (1970). Survey of capital budgeting: Theory and practice. *Journal of Finance* 25, 349-369.

[22] Menezes, C.F., C. Geiss and J. Tressler (1980). Increasing downside risk. *American Economic Review* 70, 921-932.

[23] Muliere, P. and M. Scarsini (1989). A note on stochastic dominance and inequality measures. *Journal of Economic Theory* 49, 314-323.

[24] Noussair, C.N., S.T. Trautmann and G. van de Kuilen (2014). Higher order risk attitudes, demographics, and financial decisions. *Review of Economic Studies* 81, 325-355.

[25] Prelec, D. (1998). The probability weighting function. *Econometrica* 66, 497-527.

[26] Quiggin, J. (1982). A theory of anticipated utility. *Journal of Economic Behaviour and Organization* 3, 323-343.

[27] Roëll, A. (1987). Risk aversion in Quiggin and Yaari's rank-order model of choice under uncertainty. *The Economic Journal* 97, 143-159.

[28] Rothschild, M. and J.E. Stiglitz (1970). Increasing risk: I. A definition. *Journal of Economic Theory* 2, 225-243.

[29] Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society* 97, 253-261.

[30] Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica* 57, 571-587.

[31] Tversky, A. and D. Kahneman (1992). Advances in prospect theory: Cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5, 297-323.

[32] Wang, S.S. and V.R. Young (1998). Ordering risks: Expected utility theory versus Yaari's dual theory of risk. *Insurance: Mathematics and Economics* 22, 145-161.

[33] Whitmore, G. (1970). Third order stochastic dominance. *American Economic Review* 50, 457-459.

[34] Wu, G. and R. Gonzalez (1996). Curvature of the probability weighting function. *Management Science* 42, 1676-1690.

[35] Wu, G. and R. Gonzalez (1998). Common consequence effects in decision making under risk. *Journal of Risk and Uncertainty* 16, 115-139.

[36] Yaari, M.E. (1986). Univariate and multivariate comparisons of risk aversion: a new approach. In: Heller, W.P., R.M. Starr and D.A. Starrett (Eds.). *Uncertainty, Information, and Communication.* Essays in honor of Kenneth J. Arrow, Volume III, pp. 173-188, 1st Ed., Cambridge University Press, Cambridge.

[37] Yaari, M.E. (1987). The dual theory of choice under risk. *Econometrica* 55, 95-115.