Supplement to "Systemic Risk: Conditional Distortion Risk Measures"

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Abstract

This online supplementary material provides several numerical illustrations of our main comparison results. For context, notation, and definitions we refer to the paper.

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1 Numerical Examples

We provide some numerical examples to illustrate our main findings. Based on our results developed in the paper, the choice of the d.f.'s and distortion functions of X and X' can be arbitrary since we only need the relation between u_g and $u_{g'}/u_{\tilde{g}}$. Therefore, we do not specify the explicit d.f.'s of X and X' in most of our examples. We shall provide illustrations of our main results both for positive and negative dependence structures, which are represented by the Gumbel copula and the Farlie-Gumbel-Morgenstern (FGM) copula, respectively.

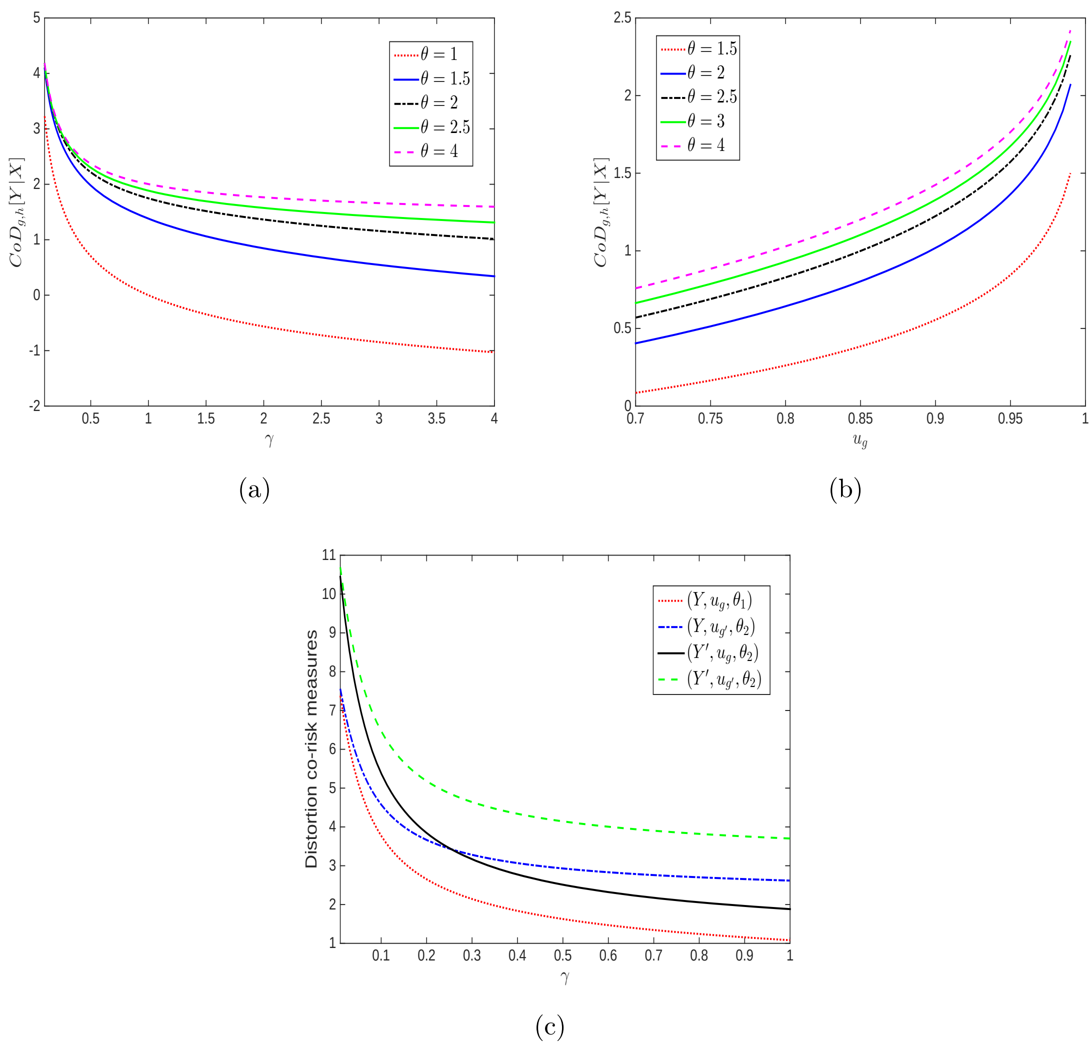


Figure 1: (a) Plot of $\mathrm{CoD}_{g,h}[Y|X]$ on $\gamma \in [0.1, 4]$ for different values of θ. (b) Plot of $\mathrm{CoD}_{g,h}[Y|X]$ on $u_g \in [0.7, 0.99]$ for different values of θ. (c) Plot of CoD-risk measures on $\gamma \in (0, 1]$ under different settings of d.f., threshold quantile, and dependence parameter.

1.1 The Gumbel Copula

The Gumbel copula is defined as

$$C_\theta(u,v) = \exp\left(-\left((-\log u)^\theta + (-\log v)^\theta\right)^{1/\theta}\right), \quad \theta \geq 1.$$

It corresponds to the independence copula when $\theta = 1$, and to the comonotonic copula when $\theta = +\infty$. It can be inferred from Wei and Hu (2002) that $C_\theta \prec C_{\theta'}$ if $\theta \leq \theta'$. Besides, C_θ is PDS for all $\theta \geq 1$. Interested readers are referred to Joe (1997) and Nelsen (2007) for more discussions.

Example 1.1 (CoD-risk measures)**.** *Assume that Y has a standard normal d.f. and $u_g = 0.95$ for some chosen d.f. of X and distortion function g. Let $h(p) = p^\gamma$ for $\gamma > 0$. Note that $h(p)$ is decreasing in γ for any $p \in [0,1]$.*

(a) For different values of the dependence parameter $\theta = 1, 1.5, 2, 2.5, 4$, we plot the values of $\mathrm{CoD}_{g,h}[Y|X]$ for $\gamma > 0$ in Figure 1(a). It is readily apparent that the CoD-risk measure decreases as the distortion function of Y gets smaller (i.e., γ gets larger) for fixed dependence parameter θ, and it increases when the positive dependence gets stronger (i.e., θ gets larger). This illustrates the result of Theorem 4.1.

(b) For different values of the dependence parameter $\theta = 1.5, 2, 2.5, 3, 4$, we plot the values of $\mathrm{CoD}_{g,h}[Y|X]$ as u_g varies from 0.7 to 0.99 in Figure 1(b), from which we observe that the CoD-risk measure increases as the threshold quantile u_g gets larger for fixed dependence parameter θ, and it increases when the positive dependence gets stronger (i.e., θ gets larger). Therefore, the theoretical finding in Theorem 4.4(i) is verified.

(c) Consider $Y \sim N(0,1)$ and $Y' \sim N(0,2)$ such that $Y \leq_{\mathrm{icx}} Y'$ but $Y \nleq_{\mathrm{st}} Y'$. Assume that $\theta_1 = 2$, $\theta_2 = 4$, $u_g = 0.8$, and $u_{g'} = 0.99$. Figure 1(c) gives the plots of $\mathrm{CoD}_{g,h}[Y|X]$, $\mathrm{CoD}_{g',h}[Y|X']$, $\mathrm{CoD}_{g,h}[Y'|X]$, and $\mathrm{CoD}_{g',h}[Y'|X']$ for different values of $\gamma \in (0,1]$, which implies that $h(p)$ is increasing and concave on $p \in [0,1]$. It is readily apparent that these four types of CoD-risk measures become smaller as γ increases, i.e., as the distortion function becomes smaller. Moreover, for any fixed $\gamma \in (0,1]$, we have

$$\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g',h}[Y|X'] \leq \mathrm{CoD}_{g',h}[Y'|X'],$$

$$\mathrm{CoD}_{g,h}[Y|X] \leq \mathrm{CoD}_{g,h}[Y'|X] \leq \mathrm{CoD}_{g',h}[Y'|X'],$$

while $\mathrm{CoD}_{g',h}[Y|X']$ and $\mathrm{CoD}_{g,h}[Y'|X]$ cannot be compared. These observations validate the results of Theorem 4.11(i).

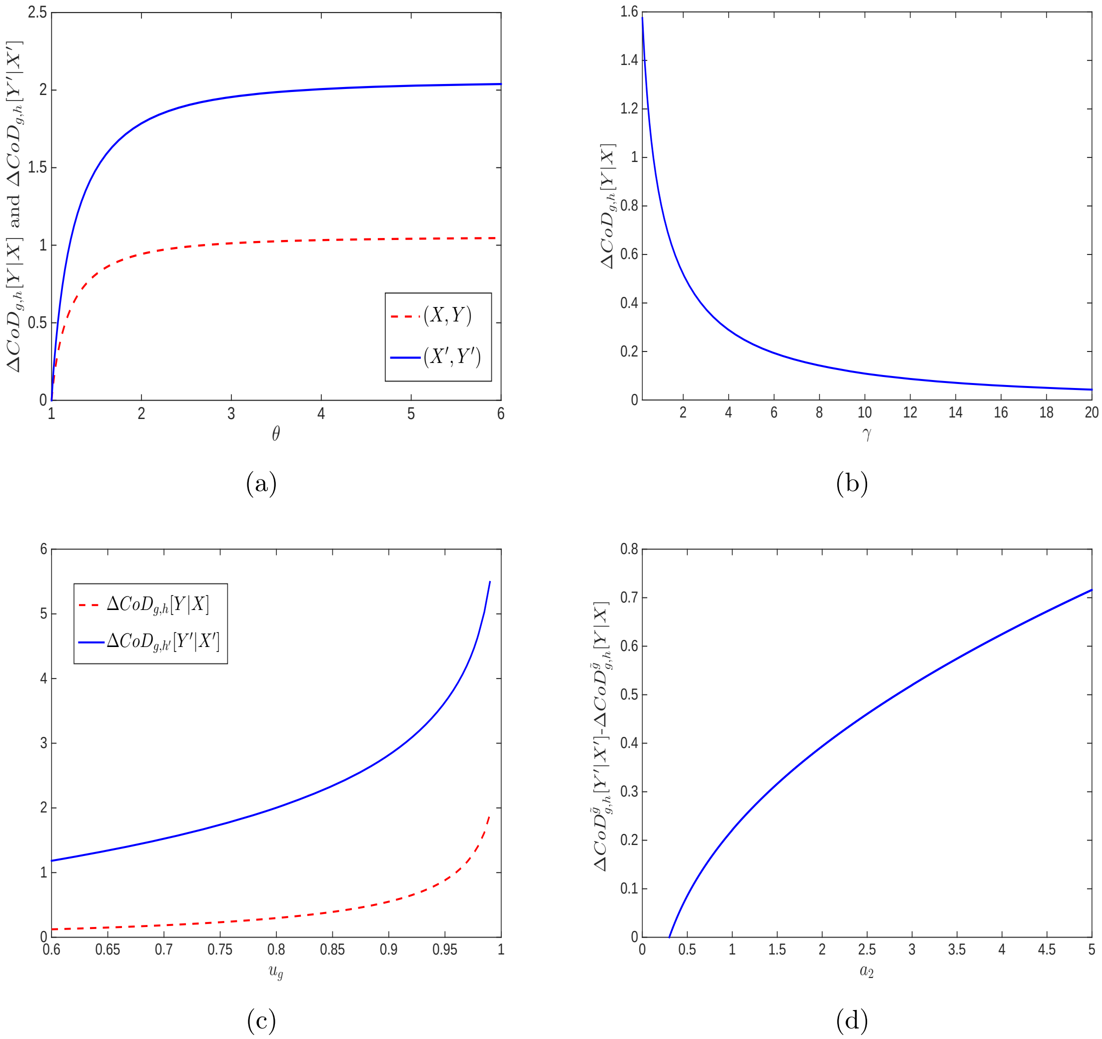


Figure 2: (a) Plot of $\Delta\text{CoD}_{g,h}[Y|X]$ and $\Delta\text{CoD}_{g,h}[Y'|X']$ on $\theta \geq 1$. (b) Plot of $\Delta\text{CoD}_{g,h}[Y|X]$ on $\gamma > 0$. (c) Plot of $\Delta\text{CoD}_{g,h}[Y|X]$ and $\Delta\text{CoD}_{g,h'}[Y'|X']$ on $u_g \in [0.6, 0.99]$. (d) Plot of $\Delta\text{CoD}^{\tilde{g}}_{g,h}[Y'|X'] - \Delta\text{CoD}^{\tilde{g}}_{g,h}[Y|X]$ for different values of the shape parameter $a_2 \geq a_1$.

The next example supports our comparison results for the distortion risk contribution measures.

Example 1.2 (Distortion risk contribution measures)**.** *In this example, we assume that the distortion functions applied to Y and Y' are of the form of a power function.*

(a) Suppose that $Y \sim \Gamma(a_1, b_1)$ and $Y' \sim \Gamma(a_2, b_2)$ with $(a_1, b_1) = (0.3, 1)$ and $(a_2, b_2) = (2, 1)$. Thus, it holds that $Y \leq_{\text{disp}} Y'$. Let $u_g = 0.8$ and $h(p) = p^{0.4}$, for $p \in [0, 1]$. Figure 2(a) displays the plots of $\Delta\text{CoD}_{g,h}[Y|X]$ and $\Delta\text{CoD}_{g,h}[Y'|X']$ on $\theta \geq 1$, from which one can observe that $\Delta\text{CoD}_{g,h}[Y|X] \leq \Delta\text{CoD}_{g,h}[Y'|X']$ for any fixed θ, and both of them are increasing with respect to '$\prec$'. This supports the result of Theorem 5.2(i).

(b) Let $Y \sim \Gamma(0.2, 1)$, $u_g = 0.9$, and $\theta = 2$. It is clear that Y is DFR. The value of $\Delta\text{CoD}_{g,h}[Y|X]$ is plotted in Figure 2(b) for different distortion functions applied to Y. It is straightforward to observe that $\Delta\text{CoD}_{g,h}[Y|X]$ is decreasing with respect to γ, which verifies Theorem 5.5(i).

(c) Let $h(p) = p^{\gamma_1}$, $h'(p) = p^{\gamma_2}$, C with parameter θ_1 and C' with parameter θ_2. Set $\gamma_1 = 3$, $\gamma_2 = 2$, $\theta_1 = 2$, $\theta_2 = 3$, $Y \sim \Gamma(0.2, 1)$, and $Y' \sim \Gamma(2, 1)$. Figure 2(c) plots $\Delta\text{CoD}_{g,h}[Y|X]$ and $\Delta\text{CoD}_{g,h'}[Y'|X']$ on $u_g \in [0.6, 0.99]$. We observe that both $\Delta\text{CoD}_{g,h}[Y|X]$ and $\Delta\text{CoD}_{g,h'}[Y'|X']$ are increasing with respect to u_g, and $\Delta\text{CoD}_{g,h}[Y|X] \leq \Delta\text{CoD}_{g,h'}[Y'|X']$ for any fixed u_g, which validates the result of Theorem 5.8.

(d) Assume that $u_g = 0.9$, $u_{\tilde{g}} = 0.8$, $h(p) = p^2$, $\theta = 2$, $Y \sim \Gamma(a_1, 1)$, and $Y' \sim \Gamma(a_2, 1)$ with $a_2 > 0$. The difference function between $\Delta\text{CoD}^{\tilde{g}}_{g,h}[Y'|X']$ and $\Delta\text{CoD}^{\tilde{g}}_{g,h}[Y|X]$ is plotted in Figure 2(d), which is always negative for all $a_2 \geq a_1 = 0.3$. Thus, the result of Theorem 5.9 is validated.

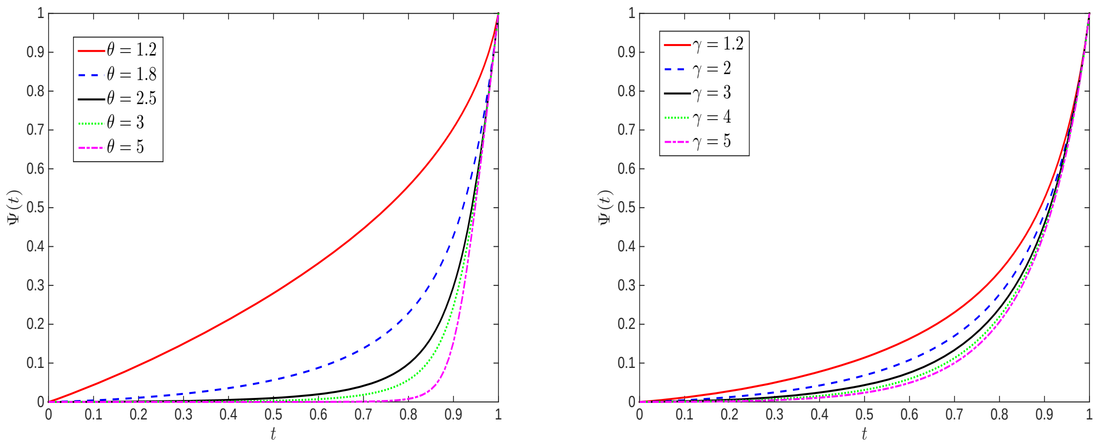
Next, we present an example to illustrate the condition in Theorem 5.13.

Example 1.3. *Assume that $h(p) = 1 - (1-p)^\gamma$ for $\gamma > 1$. Let C be the Gumbel copula with dependence parameter $\theta > 1$. It is easy to verify that $h(p)$ is concave and $\overline{h}(p) = p^\gamma$. Observe that*

$$\Psi(t) := \overline{h}(A(\overline{h}^{-1}(t))) = \left[\frac{t^{\frac{1}{\gamma}} - C(u_g, t^{\frac{1}{\gamma}})}{1 - u_g}\right]^\gamma.$$

(a) Set $u_g = 0.9$ and $\gamma = 1.1$. Figure 3(a) plots $\Psi(t)$ on $t \in [0, 1]$ under different values of $\theta = 1.2, 1.8, 2.5, 3, 5$, which indicates the convexity of $\Psi(t)$.

(b) Set $u_g = 0.9$ and $\theta = 1.5$. Figure 3(b) plots $\Psi(t)$ on $t \in [0, 1]$ under different values of $\gamma = 1.2, 2, 3, 4, 5$, from which one can observe the convexity of $\Psi(t)$.



(a) (b)

Figure 3: (a) Plot of $\Psi(t)$ on $t \in [0, 1]$ for different values of θ. (b) Plot of $\Psi(t)$ on $t \in [0, 1]$ for different values of γ.

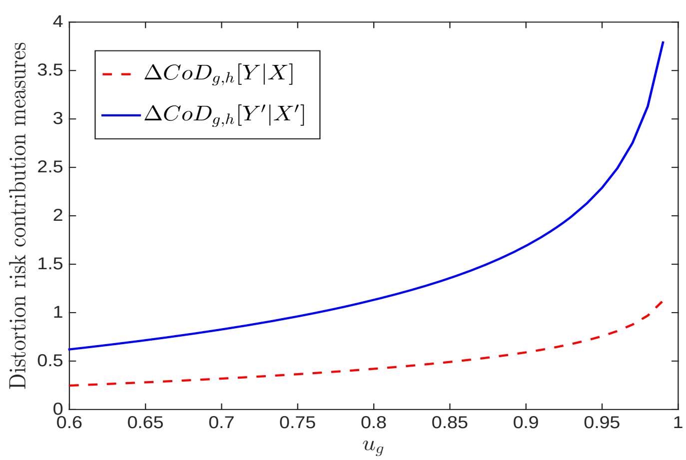


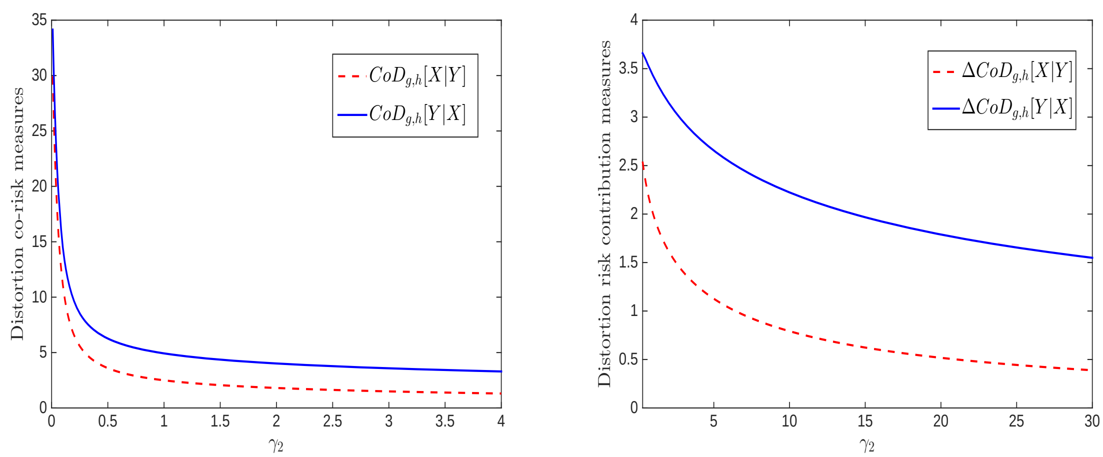
Figure 4: Plot of $\Delta \mathrm{CoD}_{g,h}[Y|X]$ and $\Delta \mathrm{CoD}_{g,h}[Y'|X']$ for $u_g \in [0.6, 0.99]$.

The following example illustrates Theorem 5.13.

Example 1.4. *Assume that* $h(p) = 1 - (1-p)^2$*,* $\theta = 1.5$*,* $Y \sim W(1,2)$*, and* $Y \sim W(1,1)$*. Clearly, it holds that* $Y \leq_{\rm ew} Y'$ *but* $Y \nleq_{\rm disp} Y'$ *nor* $Y \ngeq_{\rm disp} Y'$ *(see Example 24 in Sordo et al., 2018). As displayed in Figure 4,* $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g,h}[Y'|X']$ *for* $u_g \in [0.6, 0.99]$*, which shows the effectiveness of Theorem 5.13.*

Next, we present a numerical example to show the effectiveness of Theorem 6.1.

Example 1.5. *Let* C *be the Gumbel copula with dependence parameter* $\theta = 2$*. Assume that* $g(t) = t^{0.3}$*,* $X \sim \Gamma(0.5, 1)$*,* $Y \sim \Gamma(1.5, 1)$*, and* $h(p) = p^{\gamma_2}$ *for* $\gamma_2 > 0$*. It is easy to verify that* $X \leq_{\rm st} Y$ *and* $X \leq_{\rm disp} Y$*. Moreover, one can calculate that* $u_g^X = 0.9714 > u_g^Y = 0.9599$*. Figure 5(a) displays* $\mathrm{CoD}_{g,h}[Y|X]$ *and* $\mathrm{CoD}_{g,h}[X|Y]$ *for* $\gamma_2 > 0$*, and Figure 5(b) plots* $\Delta\mathrm{CoD}_{g,h}[Y|X]$ *and* $\Delta\mathrm{CoD}_{g,h}[X|Y]$ *for* $\gamma_2 > 0$*. Obviously,* $\mathrm{CoD}_{g,h}[Y|X] \geq \mathrm{CoD}_{g,h}[X|Y]$ *and* $\Delta\mathrm{CoD}_{g,h}[Y|X] \geq \Delta\mathrm{CoD}_{g,h}[X|Y]$ *for* $\gamma_2 > 0$*. Therefore, the results of Theorem 6.1(i) and Theorem 6.1(v) are supported.*



(a) (b)

Figure 5: (a) Plot of $\mathrm{CoD}_{g,h}[Y|X]$ and $\mathrm{CoD}_{g,h}[X|Y]$ for $\gamma_2 > 0$. (b) Plot of $\Delta\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta\mathrm{CoD}_{g,h}[X|Y]$ for $\gamma_2 > 0$.

1.2 The Farlie-Gumbel-Morgenstern Copula

The Farlie-Gumbel-Morgenstern (FGM) copula is defined as

$$C_\alpha(u,v) = uv\left[1 + \alpha(1-u)(1-v)\right], \quad -1 \leq \alpha \leq 1.$$

If $\theta = 0$, then C_θ reduces to the independence copula. Furthermore, $C_\alpha(u,v)$ is RR_2 [TP_2] for $\alpha \in [-1, 0)$ [$\alpha \in [0,1]$] and $\alpha_1 \leq \alpha_2$ implies that $C_{\alpha_1} \prec C_{\alpha_2}$. For more details on its properties, we refer to Joe (1997) and Nelsen (2007).

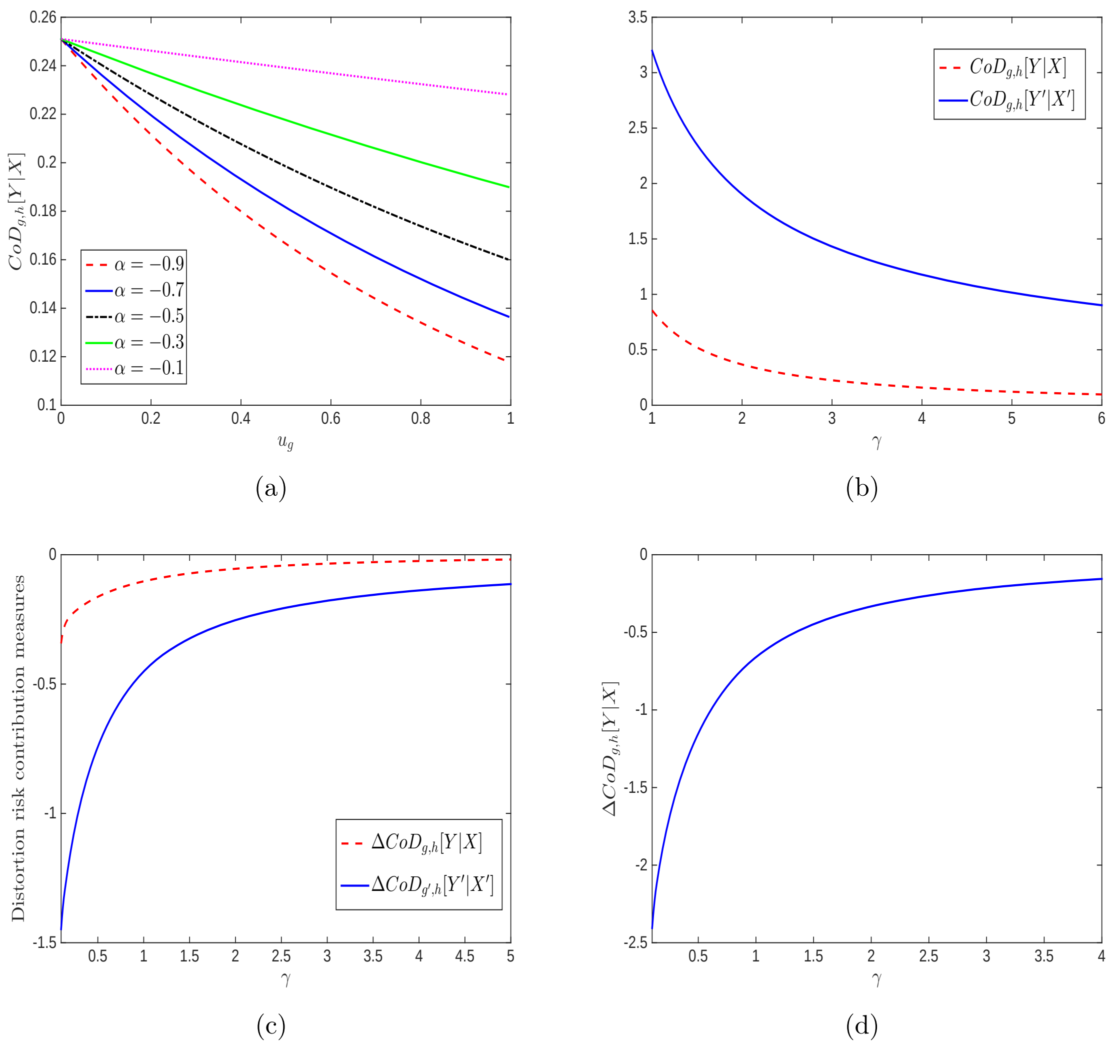
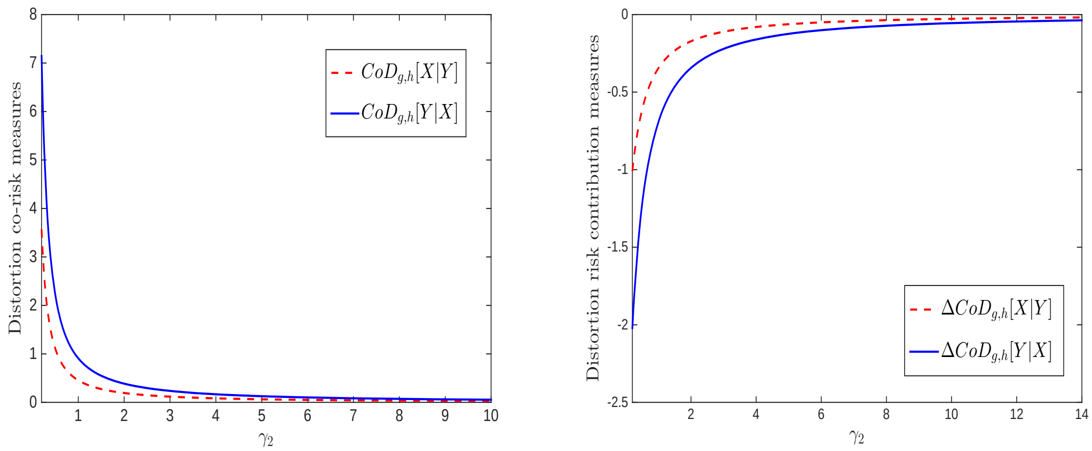


Figure 6: (a) Plot of $\mathrm{CoD}_{g,h}[Y|X]$ for $u_g \in [0,1)$. (b) Plot of $\Delta\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta\mathrm{CoD}_{g,h}[Y'|X']$ for $\gamma \geq 1$. (c) Plot of $\Delta\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta\mathrm{CoD}_{g',h}[Y'|X']$ for $\gamma > 0$. (d) Plot of $\Delta\mathrm{CoD}_{g,h}[Y|X]$ for $\gamma > 0$.

The following examples show the effectiveness of Theorem 4.4(ii), Theorem 4.7(ii), Theorem 5.3(ii), Theorem 5.5(ii), and Theorem 6.1 under the negative dependence characterized by the FGM copula.



(a) (b)

Figure 7: (a) Plot of $\mathrm{CoD}_{g,h}[Y|X]$ and $\mathrm{CoD}_{g,h}[X|Y]$ for $\gamma_2 > 0$. (b) Plot of $\Delta\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta\mathrm{CoD}_{g,h}[X|Y]$ for $\gamma_2 > 0$.

Example 1.6. *(a) Set $Y \sim \Gamma(0.8, 2)$ and $h(p) = p^5$ for $p \in [0, 1]$. Figure 6(a) displays $\mathrm{CoD}_{g,h}[Y|X]$ on $u_g \in [0, 1)$ for different values of the dependence parameter $\alpha = -0.9, -0.7, -0.5, -0.3, -0.1$. One readily observes that $\mathrm{CoD}_{g,h}[Y|X]$ is decreasing with respect to u_g for any fixed α, while it is increasing in α for any fixed u_g. This agrees with the result of Theorem 4.4(ii).*

(b) Set $Y \sim \Gamma(0.8, 2)$, $Y' \sim \Gamma(1.8, 2)$, $u_g = 0.95$, $\alpha_1 = -0.9$, and $\alpha_2 = -0.3$. Let $h(p) = p^\gamma$ for $\gamma \geq 1$ and $p \in [0, 1]$, which means that h is increasing and convex. The values of $\Delta\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta\mathrm{CoD}_{g,h}[Y'|X']$ are plotted in Figure 6(b) for $\gamma \geq 1$, from which it is clear that $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g,h}[Y'|X']$ for $\gamma \geq 1$. Thus, the result of Theorem 4.7(ii) is validated.

(c) Suppose that $Y \sim \Gamma(0.6, 1)$, $Y' \sim \Gamma(1.2, 1)$, $u_g = 0.95$, $u_{g'} = 0.9$, $\alpha = -0.3$, and $\alpha' = -0.9$. It is plain that $Y \leq_{\mathrm{disp}} Y'$, $u_g \geq u_{g'}$, and $C' \prec C$. As observed from Figure 6(c), $\Delta\mathrm{CoD}_{g,h}[Y|X] \geq \Delta\mathrm{CoD}_{g',h}[Y'|X']$ for $\gamma > 0$, which illustrates Theorem 5.3(ii).

(d) Set $\alpha = -0.8$, $Y \sim \Gamma(0.8, 2)$, and $u_g = 0.95$. Thus, Y is DFR. Let $h(p) = p^\gamma$ for $\gamma > 0$. As shown in Figure 6(d), the value of $\Delta\mathrm{CoD}_{g,h}[Y|X]$ is increasing with respect to $\gamma > 0$, which validates the theoretical finding of Theorem 5.5(ii).

Finally, a numerical example is provided to show the effectiveness of Theorem 6.1 under the case of negative dependence.

Example 1.7. *Let C be the FGM copula with dependence parameter $\alpha = -0.8$. Assume that $g(p) = p^{0.2}$, $X \sim \Gamma(0.8, 1)$, $Y \sim \Gamma(0.8, 2)$, and $h(p) = p^{\gamma_2}$ for $\gamma_2 > 0$. It is easy to verify that $X \leq_{\rm hr} Y$ and thus $X \leq_{\rm st\ [disp]} Y$ since both X and Y are DFR. Moreover, one can calculate that $u_g^X = u_g^Y = 0.9937$. Figure 7(a) displays $\mathrm{CoD}_{g,h}[Y|X]$ and $\mathrm{CoD}_{g,h}[X|Y]$ for $\gamma_2 > 0$, and Figure 7(b) plots $\Delta\mathrm{CoD}_{g,h}[Y|X]$ and $\Delta\mathrm{CoD}_{g,h}[X|Y]$ for $\gamma_2 > 0$. Note that $\mathrm{CoD}_{g,h}[Y|X] \geq \mathrm{CoD}_{g,h}[X|Y]$ while $\Delta\mathrm{CoD}_{g,h}[Y|X] \leq \Delta\mathrm{CoD}_{g,h}[X|Y]$ for all $\gamma_2 > 0$, and thus the results of Theorems 6.1(ii) and 6.1(iv) are validated.*

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