Supplement to “Two-Sample Testing for Tail Copulas with an Application to Equity Indices”

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Abstract

This text serves as an appendix to the paper “Two-Sample Testing for Tail Copulas with an Application to Equity Indices.” For context, notation and definitions, see that paper. We provide the proofs of Theorems 3.1, 4.1 and 5.1.

Proof of Theorem 3.1

By Skorohod's representation theorem, there is a probability space where probabilistically equivalent versions of all the random elements in Assumption A1 are defined, those in (9) independent of those in (10), and the convergences (9) and (10) hold in probability. All statements in this proof should be understood as statements about random elements in this probability space.

Given $(x, y) \in [0, \infty)^2$, let us define points $(\widehat{x}, \widehat{y})$ and $(\widehat{x}', \widehat{y}')$ by

$$\begin{aligned}
\widehat{x} &= \left[\left(1 + \gamma_1\left(\frac{x^{-\widehat{\gamma}_1} - 1}{\widehat{\gamma}_1} \cdot \frac{\widehat{a}_1}{a_1} + \frac{\widehat{b}_1 - b_1}{a_1}\right)\right) \vee 0\right]^{-1/\gamma_1}, \\
\widehat{y} &= \left[\left(1 + \gamma_2\left(\frac{y^{-\widehat{\gamma}_2} - 1}{\widehat{\gamma}_2} \cdot \frac{\widehat{a}_2}{a_2} + \frac{\widehat{b}_2 - b_2}{a_2}\right)\right) \vee 0\right]^{-1/\gamma_2}, \\
\widehat{x}' &= \left[\left(1 + \gamma_1'\left(\frac{x^{-\widehat{\gamma}_1'} - 1}{\widehat{\gamma}_1'} \cdot \frac{\widehat{a}_1'}{a_1'} + \frac{\widehat{b}_1' - b_1'}{a_1'}\right)\right) \vee 0\right]^{-1/\gamma_1'}, \\
\widehat{y}' &= \left[\left(1 + \gamma_2'\left(\frac{y^{-\widehat{\gamma}_2'} - 1}{\widehat{\gamma}_2'} \cdot \frac{\widehat{a}_2'}{a_2'} + \frac{\widehat{b}_2' - b_2'}{a_2'}\right)\right) \vee 0\right]^{-1/\gamma_2'}.
\end{aligned}$$

It follows from Lemma 1.1 in the Appendix of Can et al. (2015) that

$$\begin{aligned}
&\sup_{x \in [\delta, T]} \left|\sqrt{k}(\widehat{x} - x) - \left[f(x, \gamma_1)A_1 + g(x, \gamma_1)B_1 + h(x, \gamma_1)\Gamma_1\right]\right| \xrightarrow{P} 0, \\
&\sup_{y \in [\delta, T]} \left|\sqrt{k}(\widehat{y} - y) - \left[f(y, \gamma_2)A_2 + g(y, \gamma_2)B_2 + h(y, \gamma_2)\Gamma_2\right]\right| \xrightarrow{P} 0, \\
&\sup_{x \in [\delta, T]} \left|\sqrt{k'}(\widehat{x}' - x) - \left[f(x, \gamma_1')A_1' + g(x, \gamma_1')B_1' + h(x, \gamma_1')\Gamma_1'\right]\right| \xrightarrow{P} 0, \\
&\sup_{y \in [\delta, T]} \left|\sqrt{k'}(\widehat{y}' - y) - \left[f(y, \gamma_2')A_2' + g(y, \gamma_2')B_2' + h(y, \gamma_2')\Gamma_2'\right]\right| \xrightarrow{P} 0.
\end{aligned} \tag{S.1}$$

Now, let $\widehat{R}_n$ and T_n be as defined in (7) and (8), respectively, and let $\widehat{R}'_{n'}$ and $T'_{n'}$ be their analogues constructed from the second sample. Note that the probability of the event

$$\{\widehat{R}_n(x, y) = T_n(\widehat{x}, \widehat{y}) \text{ and } \widehat{R}'_{n'}(x, y) = T'_{n'}(\widehat{x}', \widehat{y}') \text{ for all } (x, y) \in [\delta, T]^2\} \tag{S.2}$$

tends to 1 as $n, n' \to \infty$. Hence, instead of $\eta_{n,n'}$, it will suffice to show convergence for

$\eta^*_{n,n'}(x,y) := \sqrt{\kappa}[T_n(\widehat{x},\widehat{y}) - T'_{n'}(\widehat{x}',\widehat{y}')]$, which we decompose as follows:

$$\begin{aligned}\eta^*_{n,n'}(x,y) &= \sqrt{\kappa}[T_n(\widehat{x},\widehat{y}) - R_n(\widehat{x},\widehat{y})] + \sqrt{\kappa}[R_n(\widehat{x},\widehat{y}) - R(\widehat{x},\widehat{y})] \\ &\quad + \sqrt{\kappa}[R(\widehat{x},\widehat{y}) - R(x,y)] - \sqrt{\kappa}[T'_{n'}(\widehat{x}',\widehat{y}') - R'_{n'}(\widehat{x}',\widehat{y}')] \\ &\quad - \sqrt{\kappa}[R'_{n'}(\widehat{x}',\widehat{y}') - R(\widehat{x}',\widehat{y}')] - \sqrt{\kappa}[R(\widehat{x}',\widehat{y}') - R(x,y)] \\ &=: \eta^*_{1n}(x,y) + \eta^*_{2n}(x,y) + \eta^*_{3n}(x,y) - \eta^*_{4n'}(x,y) - \eta^*_{5n'}(x,y) - \eta^*_{6n'}(x,y).\end{aligned}$$

The in-probability convergence (9), (S.1) and the continuity of V_R yield

$$\sup_{(x,y)\in[\delta,T]^2} \left|\eta^*_{1n}(x,y) - \sqrt{c}V_R(x,y)\right| \xrightarrow{P} 0. \tag{S.3}$$

From Assumption A3 and (S.1) it also follows that

$$\sup_{(x,y)\in[\delta,T]^2} \left|\eta^*_{2n}(x,y)\right| \xrightarrow{P} 0. \tag{S.4}$$

Moreover, from the Mean Value Theorem we know that

$$\eta^*_{3n}(x,y) = \sqrt{\kappa}[R^{(1)}(\breve{x},\breve{y})(\widehat{x} - x) + R^{(2)}(\breve{x},\breve{y})(\widehat{y} - y),$$

for some $(\breve{x},\breve{y})$ lying on the line segment connecting (x,y) and $(\widehat{x},\widehat{y})$. The convergence (S.1) in combination with Assumption A2 now yields that

$$\begin{aligned}\sup_{(x,y)\in[\delta,T]^2} \Big|\eta^*_{3n}(x,y) &- \sqrt{c}R^{(1)}(x,y)\big[f(x,\gamma_1)A_1 + g(x,\gamma_1)B_1 + h(x,\gamma_1)\Gamma_1\big] \\ &- \sqrt{c}R^{(2)}(x,y)\big[f(y,\gamma_2)A_2 + g(y,\gamma_2)B_2 + h(y,\gamma_2)\Gamma_2\big]\Big| \xrightarrow{P} 0.\end{aligned} \tag{S.5}$$

The analogues of (S.3), (S.4), (S.5) for $\eta^*_{4n'}$, $\eta^*_{5n'}$, $\eta^*_{6n'}$ are proved along similar lines. Combining these six results completes the proof.

Proof of Theorem 4.1

This result essentially follows from the general martingale transformation result in Theorem 3.1 of Can et al. (2015). Instead of arbitrary Borel sets $B \subset [\delta,T]^2$ considered therein, we consider rectangles $[\delta, \delta + x] \times [\delta, \delta + y]$ for $0 \le x, y \le T - \delta$. Furthermore, we use the scanning family $A_u = [\delta, T] \times [\delta, (1-u)\delta + uT]$, $0 \le u \le 1$. Then, from (13) above under Assumptions A4–A5 confined to the true $\gamma_1, \gamma_2, \gamma_1', \gamma_2', r, r^{(1)}, r^{(2)}$, we obtain that

$$
\begin{aligned}
W_R(x,y) = & \int_\delta^{\delta+x} \int_\delta^{\delta+y} \mathrm{d}\eta(s,t) \\
& - \int_\delta^{\delta+x} \int_\delta^{\delta+y} \mathbf{q}(s,t)^\mathsf{T} \Big(\mathbf{I}_{\delta,T}^{-1}(t) \int_\delta^T \int_t^T \mathbf{q}(s',t') \, \mathrm{d}\eta(s',t') \Big) r(s,t) \, \mathrm{d}s \, \mathrm{d}t
\end{aligned}
$$

is a bivariate Wiener process on $[0,\tau]^2$ for any $\tau \in (\delta, T-\delta)$, with "time" measure $R([\delta, \delta + \cdot\,] \times [\delta, \delta + \cdot\,])$. That is, W_R is a zero-mean Gaussian process with covariance structure

$$
E[W_R(x,y) W_R(x',y')] = R([\delta, \delta + x \wedge x'] \times [\delta, \delta + y \wedge y']),
$$

for $(x,y), (x',y') \in [0,\tau]^2$. It then follows from the standard theory of multivariate Gaussian processes (see, e.g., the lemma preceding Theorem 3 in Khmaladze (1988)) that the normalized process

$$
\begin{aligned}
W(x,y) &= \int_\delta^{\delta+x} \int_\delta^{\delta+y} \frac{1}{\sqrt{r(s,t)}} \, \mathrm{d}W_R(s,t) \\
&= \int_\delta^{\delta+x} \int_\delta^{\delta+y} \frac{1}{\sqrt{r(s,t)}} \, \mathrm{d}\eta(s,t) \\
&\quad - \int_\delta^{\delta+x} \int_\delta^{\delta+y} \mathbf{q}(s,t)^\mathsf{T} \Big(\mathbf{I}_{\delta,T}^{-1}(t) \int_\delta^T \int_t^T \mathbf{q}(s',t') \, \mathrm{d}\eta(s',t') \Big) \sqrt{r(s,t)} \, \mathrm{d}s \, \mathrm{d}t
\end{aligned}
$$

is a standard bivariate Wiener process on $[0,\tau]^2$.

Proof of Theorem 5.1

First we establish the consistency of the estimator $\widehat{r}$, that is:

$$
\sup_{(x,y) \in [\delta,T]^2} |\widehat{r}(x,y) - r(x,y)| \xrightarrow{P} 0, \quad \text{as } n, n' \to \infty. \tag{S.6}
$$

Clearly, it is sufficient to show that

$$
\sup_{(x,y) \in [\delta,T]^2} |s_h(x,y) - r(x,y)| \xrightarrow{P} 0, \quad \text{as } n \to \infty.
$$

We write

$$
\begin{aligned}
&s_h(x,y) - r(x,y) \\
&\quad = \frac{1}{(2h)^2}\Big[\widehat{R}_n\big([x-h,x+h]\times[y-h,y+h]\big) - R\big([x-h,x+h]\times[y-h,y+h]\big)\Big] \\
&\qquad + \frac{1}{(2h)^2} R\big([x-h,x+h]\times[y-h,y+h]\big) - r(x,y).
\end{aligned}
$$

From (S.3), (S.4) and (S.5), the first term is $O_P(k^{-1/2}(k^{1/6})^2) = o_P(1)$ uniformly on $[\delta,T]^2$. The second term is equal to

$$
(2h)^{-2}\int_{y-h}^{y+h}\int_{x-h}^{x+h} \big(r(u,v)-r(x,y)\big)\,\mathrm{d}u\,\mathrm{d}v,
$$

which by the (uniform) continuity of r tends to 0, uniformly on $[\delta,T]^2$.

Next, we establish the consistency of $\widehat{r}^{(1)}$ and $\widehat{r}^{(2)}$, that is,

$$
\sup_{(x,y)\in[\delta,T]^2} |\widehat{r}^{(j)}(x,y) - r^{(j)}(x,y)| \xrightarrow{P} 0, \quad \text{as } n, n' \to \infty, \tag{S.7}
$$

for $j = 1, 2$. For this it is sufficient to show that

$$
\sup_{(x,y)\in[\delta,T]^2} \left| \frac{s_h(x+h,y)-s_h(x-h,y)}{2h} - r^{(1)}(x,y) \right| \xrightarrow{P} 0, \quad \text{as } n \to \infty.
$$

We have

$$
\begin{aligned}
&\frac{s_h(x+h,y)-s_h(x-h,y)}{2h} - r^{(1)}(x,y) \\
&\quad = \frac{1}{(2h)^3}\Big\{\Big[\widehat{R}_n\big([x,x+2h]\times[y-h,y+h]\big) - \widehat{R}_n\big([x-2h,x]\times[y-h,y+h]\big)\Big] \\
&\qquad\qquad - \Big[R\big([x,x+2h]\times[y-h,y+h]\big) - R\big([x-2h,x]\times[y-h,y+h]\big)\Big]\Big\} \\
&\qquad + \frac{1}{(2h)^3}\Big[R\big([x,x+2h]\times[y-h,y+h]\big) - R\big([x-2h,x]\times[y-h,y+h]\big)\Big] \\
&\qquad - r^{(1)}(x,y).
\end{aligned}
$$

From (S.3)–(S.5) again, the first term is now $O_P(k^{-1/2}(k^{1/8})^3) = o_P(1)$ uniformly on $[\delta,T]^2$. The second term is equal to

$$
(2h)^{-2}\int_{y-h}^{y+h}\int_{x}^{x+2h} \left(\frac{r(u,v)-r(u-2h,v)}{2h} - r^{(1)}(x,y)\right)\mathrm{d}u\,\mathrm{d}v,
$$

which tends to 0, uniformly on $[\delta, T]^2$, by the Mean Value Theorem and the (uniform) continuity of $r^{(1)}$.

Now, by Theorem 3.1 and Skorohod's representation theorem, there is a probability space where probabilistically equivalent versions of $\eta_{n,n'}$ and η are defined, and these satisfy $\|\eta_{n,n'} - \eta\|_{[\delta,T]^2} \to 0$ a.s., with $\|\cdot\|_S := \sup_S |\cdot|$ for $S \subset [0,\infty)^2$. We will show that in this probability space,

$$\|W_{n,n'} - W\|_{[0,\tau]^2} \xrightarrow{P} 0, \tag{S.8}$$

with W as defined in (16). In view of Theorem 4.1, this will suffice for the proof.

Throughout the proof, we will let $A_\delta(x,y)$ denote the rectangle $[\delta, \delta+x] \times [\delta, \delta+y]$ for $(x,y) \in [0,\tau]^2$. Note that (S.8) will follow from

$$\left\| \int_{A_\delta(x,y)} \frac{1}{\sqrt{\widehat{r}(s,t)}} \, \mathrm{d}\eta_{n,n'}(s,t) - \int_{A_\delta(x,y)} \frac{1}{\sqrt{r(s,t)}} \, \mathrm{d}\eta(s,t) \right\|_{[0,\tau]^2} \xrightarrow{P} 0 \tag{S.9}$$

and

$$\begin{aligned} \Bigg\| \int_{A_\delta(x,y)} & \widehat{\mathbf{q}}(s,t)^\mathsf{T} \left(\widehat{\mathbf{I}}_{\delta,T}^{-1}(t) \int_\delta^T \int_t^T \widehat{\mathbf{q}}(s',t') \, \mathrm{d}\eta_{n,n'}(s',t') \right) \sqrt{\widehat{r}(s,t)} \, \mathrm{d}s \, \mathrm{d}t \\ & - \int_{A_\delta(x,y)} \mathbf{q}(s,t)^\mathsf{T} \left(\mathbf{I}_{\delta,T}^{-1}(t) \int_\delta^T \int_t^T \mathbf{q}(s',t') \, \mathrm{d}\eta(s',t') \right) \sqrt{r(s,t)} \, \mathrm{d}s \, \mathrm{d}t \Bigg\|_{[0,\tau]^2} \xrightarrow{P} 0. \end{aligned} \tag{S.10}$$

We will prove (S.9) first. Let $\Delta_{n,n'} := \eta_{n,n'} - \eta$. Then (S.9) will follow from

$$\left\| \int_{A_\delta(x,y)} \Delta\sigma(s,t) \, \mathrm{d}\eta(s,t) \right\|_{[0,\tau]^2} \xrightarrow{P} 0, \qquad \left\| \int_{A_\delta(x,y)} \widehat{\sigma}(s,t) \, \mathrm{d}\Delta_{n,n'}(s,t) \right\|_{[0,\tau]^2} \xrightarrow{P} 0. \tag{S.11}$$

Applying bivariate integration by parts (see e.g., Henstock (1973), Theorem 3) to the first integral term in (S.11), we obtain the following bound:

$$\begin{aligned} \left| \int_{A_\delta(x,y)} \Delta\sigma(s,t) \, \mathrm{d}\eta(s,t) \right| &\le \sum_{(u,v) \in \mathcal{V}_\delta(x,y)} \big| \Delta\sigma(u,v) \, \eta(u,v) \big| + \|\eta\|_{A_\delta(x,y)} V^{\mathrm{HK}}_{A_\delta(x,y)}(\Delta\sigma) \\ &\le \|\eta\|_{[\delta,T]^2} \big(4\|\Delta\sigma\|_{[\delta,T]^2} + V^{\mathrm{HK}}_{[\delta,T]^2}(\Delta\sigma) \big), \end{aligned} \tag{S.12}$$

where $\mathcal{V}_\delta(x,y)$ denotes the set of the four vertices of the rectangle $A_\delta(x,y)$. Now, Assumption A2 ensures that η is continuous (hence bounded) on $[\delta,T]^2$, (S.6) ensures that $|\Delta\sigma|$

is $o_P(1)$ uniformly over $[\delta, T]^2$, and A6 ensures that $V^{\mathrm{HK}}_{[\delta,T]^2}(\Delta\sigma)$ is $o_P(1)$ as well. It follows that the far right-hand side of (S.12) vanishes in probability, and the first convergence in (S.11) is proved. The second convergence in (S.11) follows from a similar integration by parts argument:

$$\left| \int_{A_\delta(x,y)} \widehat{\sigma}(s,t)\,\mathrm{d}\Delta_{n,n'}(s,t) \right| \leq \|\Delta_{n,n'}\|_{[\delta,T]^2} \big(4\|\widehat{\sigma}\|_{[\delta,T]^2} + V^{\mathrm{HK}}_{[\delta,T]^2}(\widehat{\sigma})\big),$$

where the right-hand side is $o_P(1)$ since $\|\Delta_{n,n'}\|_{[\delta,T]^2}$ is $o_P(1)$ and $\|\widehat{\sigma}\|_{[\delta,T]^2}$ as well as $V^{\mathrm{HK}}_{[\delta,T]^2}(\widehat{\sigma})$ are $O_P(1)$ terms.

We have thus established (S.9), and it remains to prove (S.10). For ease of notation, we let

$$\begin{aligned}
H(s,t) &= \mathbf{q}(s,t)^{\mathsf{T}} \mathbf{I}^{-1}_{\delta,t}(t) \int_\delta^T \int_t^T \mathbf{q}(s',t')\,\mathrm{d}\eta(s',t'),\\
H_{n,n'}(s,t) &= \mathbf{q}(s,t)^{\mathsf{T}} \mathbf{I}^{-1}_{\delta,T}(t) \int_\delta^T \int_t^T \mathbf{q}(s',t')\,\mathrm{d}\eta_{n,n'}(s',t'),\\
\widehat{H}(s,t) &= \widehat{\mathbf{q}}(s,t)^{\mathsf{T}} \widehat{\mathbf{I}}^{-1}_{\delta,T}(t) \int_\delta^T \int_t^T \widehat{\mathbf{q}}(s',t')\,\mathrm{d}\eta(s',t'),\\
\widehat{H}_{n,n'}(s,t) &= \widehat{\mathbf{q}}(s,t)^{\mathsf{T}} \widehat{\mathbf{I}}^{-1}_{\delta,T}(t) \int_\delta^T \int_t^T \widehat{\mathbf{q}}(s',t')\,\mathrm{d}\eta_{n,n'}(s',t').
\end{aligned}$$

Then (S.10) can be written succinctly as

$$\left\| \int_{A_\delta(x,y)} \Big(\widehat{H}_{n,n'}(s,t)\sqrt{\widehat{r}(s,t)} - H(s,t)\sqrt{r(s,t)} \Big)\,\mathrm{d}s\,\mathrm{d}t \right\|_{[0,\tau]^2} \xrightarrow{P} 0,$$

which can be proved by showing

$$\big\|H\big(\sqrt{\widehat{r}} - \sqrt{r}\big)\big\|_{A_\delta(\tau,\tau)} \xrightarrow{P} 0, \qquad \big\|\big(\widehat{H}_{n,n'} - H\big)\sqrt{\widehat{r}}\,\big\|_{A_\delta(\tau,\tau)} \xrightarrow{P} 0. \tag{S.13}$$

The first convergence in (S.13) follows easily from the continuity (hence boundedness) of H over $A_\delta(\tau,\tau)$ and (S.6). As for the second convergence in (S.13), since $\|\sqrt{\widehat{r}}\|_{A_\delta(\tau,\tau)} = O_P(1)$, we need to show that $\|\widehat{H}_{n,n'} - H\|_{A_\delta(\tau,\tau)} \xrightarrow{P} 0$. We will do this by proving

$$\|H_{n,n'} - H\|_{A_\delta(\tau,\tau)} \xrightarrow{P} 0, \qquad \|\widehat{H}_{n,n'} - H_{n,n'}\|_{A_\delta(\tau,\tau)} \xrightarrow{P} 0. \tag{S.14}$$

Consider the first convergence in (S.14). We have

$$\|H_{n,n'} - H\|_{A_\delta(\tau,\tau)} = \left\| \mathbf{q}(s,t)^\mathsf{T} \mathbf{I}_{\delta,T}^{-1}(t) \int_\delta^T \int_t^T \mathbf{q}(s',t')\, \mathrm{d}\Delta_{n,n'}(s',t') \right\|_{A_\delta(\tau,\tau)},$$

with $\Delta_{n,n'} = \eta_{n,n'} - \eta$, as before. The term $|\mathbf{q}(s,t)^\mathsf{T}\mathbf{I}_{\delta,T}^{-1}(t)|$ is component-wise bounded on $A_\delta(\tau,\tau)$ by continuity, so we need to show that

$$\sup_{t\in[\delta,\delta+\tau]} \left| \int_\delta^T \int_t^T q_i(s',t')\, \mathrm{d}\Delta_{n,n'}(s',t') \right| \xrightarrow{P} 0, \qquad i = 1,\dots,8. \tag{S.15}$$

Applying integration by parts as before, we obtain

$$\left| \int_\delta^T \int_t^T q_i(s',t')\, \mathrm{d}\Delta_{n,n'}(s',t') \right| \le \|\Delta_{n,n'}\|_{[\delta,T]^2} \big(4\|q_i\|_{[\delta,T]^2} + V^{\mathrm{HK}}_{[\delta,T]^2}(q_i)\big),$$

where the right-hand side is $o_P(1)$ since $\|\Delta_{n,n'}\|_{[\delta,T]^2} = o_P(1)$, $\|q_i\|_{[\delta,T]^2} < \infty$ by continuity, and $V^{\mathrm{HK}}_{[\delta,T]^2}(q_i) < \infty$ by virtue of Assumption A6 together with Proposition 1 of Blümlinger and Tichy (1989). Hence (S.15) is established and it remains to prove the second convergence in (S.14).

By virtue of the first convergence in (S.14), and an analogous result for $\widehat{H}_{n,n'}$ and $\widehat{H}$, it will suffice to prove $\|\widehat{H} - H\|_{A_\delta(\tau,\tau)} \xrightarrow{P} 0$. Note that

$$\begin{aligned} |\widehat{H}(s,t) - H(s,t)| \le{}& \big|\widehat{\mathbf{q}}(s,t)^\mathsf{T}\widehat{\mathbf{I}}_{\delta,T}^{-1}(t) - \mathbf{q}(s,t)^\mathsf{T}\mathbf{I}_{\delta,T}^{-1}(t)\big| \cdot \left| \int_\delta^T \int_t^T \mathbf{q}(s',t')\, \mathrm{d}\eta(s',t') \right| \\ &+ \big|\widehat{\mathbf{q}}(s,t)^\mathsf{T}\widehat{\mathbf{I}}_{\delta,T}^{-1}(t)\big| \cdot \left| \int_\delta^T \int_t^T \big(\widehat{\mathbf{q}}(s,t') - \mathbf{q}(s',t')\big)\, \mathrm{d}\eta(s',t') \right|, \end{aligned} \tag{S.16}$$

where absolute values should be interpreted component-wise, as usual. Consider the first summand on the right-hand side of (S.16). Our assumptions about the various estimators and (S.6)–(S.7) ensure that the difference $\big|\widehat{\mathbf{q}}(s,t)^\mathsf{T}\widehat{\mathbf{I}}_{\delta,T}^{-1}(t) - \mathbf{q}(s,t)^\mathsf{T}\mathbf{I}_{\delta,T}^{-1}(t)\big|$ is $o_P(1)$ uniformly over $(s,t) \in A_\delta(\tau,\tau)$. Moreover, an integration by parts argument as before yields that

$$\left| \int_\delta^T \int_t^T q_i(s',t')\, \mathrm{d}\eta(s',t') \right| \le \|\eta\|_{[\delta,T]^2} \big(4\|q_i\|_{[\delta,T]^2} + V^{\mathrm{HK}}_{[\delta,T]^2}(q_i)\big),$$

for $i = 1,\dots,8$, where the right-hand side is $O_P(1)$. So the first summand on the right-hand side of (S.16) is $o_P(1)$ uniformly over $(s,t) \in A_\delta(\tau,\tau)$. The second summand there can be handled similarly: the term $\left|\widehat{\mathbf{q}}(s,t)^{\mathsf{T}}\widehat{\mathbf{I}}_{\delta,T}^{-1}(t)\right|$ is $O_P(1)$, and integration by parts yields

$$\left|\int_\delta^T \int_t^T \Delta q_i(s',t')\,\mathrm{d}\eta(s',t')\right| \leq \|\eta\|_{[\delta,T]^2}\big(4\|\Delta q_i\|_{[\delta,T]^2} + V^{\mathrm{HK}}_{[\delta,T]^2}(\Delta q_i)\big)$$

for $i = 1, \ldots, 8$, where the right-hand side is $o_P(1)$.

Both convergences in (S.13) are thereby established, which in turn proves (S.10).

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