Dynamic robust Orlicz premia and Haezendonck-Goovaerts risk measures[☆]

Fabio Bellini[a,*], Roger J. A. Laeven[b], Emanuela Rosazza Gianin[c]

[a] *Department of Statistics and Quantitative Methods, University of Milano-Bicocca, fabio.bellini@unimib.it*
[b] *Department of Quantitative Economics, University of Amsterdam, CentER and EURANDOM, R.J.A.Laeven@uva.nl*
[c] *Department of Statistics and Quantitative Methods, University of Milano-Bicocca, emanuela.rosazza1@unimib.it*

Abstract

In this paper we extend to a dynamic setting the robust Orlicz premia and Haezendonck-Goovaerts risk measures introduced in Bellini, Laeven and Rosazza Gianin (5). We extensively analyze the properties of the resulting dynamic risk measures. Furthermore, we characterize dynamic Orlicz premia that are time-consistent, and establish some relations between the time-consistency properties of dynamic robust Orlicz premia and the corresponding dynamic robust Haezendonck-Goovaerts risk measures.

Keywords: Risk analysis, Orlicz premia, Haezendonck-Goovaerts risk measures, Time-consistency, Ambiguity averse preferences.
2010 MSC: 60E15, 62P05.
JEL: D81, G10, G20.

[☆] We are very grateful to the Associate Editor and to two anonymous referees whose useful comments led to a substantial improvement of the paper. We are also grateful to Marco Frittelli and to conference participants at the Model Uncertainty and Robust Finance Workshop in Milano (March 2018), the 10th ASF Conference in Samos (May 2018), the 29th European Conference on Operational Research in Valencia (July 2018), the 42nd AMASES Meeting in Napoli (September 2018), and the 9th General AMAMEF Conference in Paris (June 2019) for their comments and suggestions. This research was funded in part by the Netherlands Organization for Scientific Research under grant NWO Vidi (Laeven).

[*] Corresponding author.

1. Introduction

In the context of risk pricing and capital requirements, modern static theories of financial risk measurement are provided by monetary, convex and entropy convex measures of risk (Föllmer and Schied (26; 27), Frittelli and Rosazza Gianin (29), Ruszczyński and Shapiro (48), and Laeven and Stadje (38)).

Over the past two decades not only the study of static but also of dynamic theories of risk measurement has developed into a flourishing and mathematically refined area of research. The dynamic counterparts of the static theories of monetary measures of risk have been developed in Artzner et al. (2), Riedel (45), Frittelli and Rosazza Gianin (30), Detlefsen and Scandolo (22), Cheridito, Delbaen and Kupper (12), Delbaen (19), Föllmer and Penner (25), Klöppel and Schweizer (35), Cheridito and Kupper (15), among many others. We refer to Föllmer and Schied (27), Chapter 11, for an overview and many references.

A main problem in dynamic risk measurement is the consistency over time of the evaluation as well as of the resulting decisions. The notion of *recursiveness*, or Bellman's dynamic programming principle, has played a central role in the early development of the literature on the theory and application of dynamic measurement of risk; see e.g., Duffie and Epstein (23), Chen and Epstein (11), Epstein and Schneider (24) and Ruszczyński and Shapiro (49). Recursiveness is intimately related to (strong) time-consistency (even equivalent, under linear utility).

A large literature analyzes and characterizes time-consistency for the canonical theories of risk measurement. There are several approaches to characterizing/generating time-consistency properties in the literature, including the following: approaches based on mixture representations and law invariance (Weber (52), Kupper and Schachermayer (37), Delbaen, Bellini, Bignozzi and Ziegel (21)); based on dual representations, requiring m-stability (or rectangularity) of the set of generalized scenarios, or imposing the cocycle property on the penalty function (Delbaen (19), Föllmer and Penner (25), Bion-Nadal (6; 7)); based on a decomposition property of acceptance sets (Bion-Nadal (6; 7), Cheridito, Delbaen and Kupper (12)); and based on a recursive construction via generators (Cheridito and Kupper (16)). In continuous-time, in a Brownian or Brownian-Poissonian filtration, another approach consists of characterizing time-consistency via a representation of the penalty function; see Delbaen, Peng and Rosazza Gianin (20), Tang and

Wei (50), Laeven and Stadje (39), and see Krätschmer et al. (36) and its references for corresponding applications in OR.

Recently, Bellini, Laeven and Rosazza Gianin (5) introduced *robust return risk measures*, mainly for two purposes. First, to reveal and formalize the difference between risk measurement in terms of *monetary* values and in terms of *returns*. Second, to take into account in this setting ambiguity with respect to the probabilistic model P, by means of ambiguity averse preferences, specifically invoking multiple priors (Gilboa and Schmeidler (32)), variational preferences (Maccheroni, Marinacci and Rustichini (41)), or homothetic preferences (Chateauneuf and Faro (10), Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (9) and Laeven and Stadje (38)).

In particular, in Bellini, Laeven and Rosazza Gianin (5) we provided an axiomatic foundation of a canonical subclass of return risk measures, that of *Orlicz premia*, by exploiting a one-to-one correspondence between Orlicz premia and measures of (utility-based) shortfall risk. Furthermore, we defined, axiomatized and studied robustified versions of Orlicz premia and of their optimized translation-invariant extensions (Rockafellar and Uryasev (46) and Rockafellar, Uryasev and Zabarankin (47)), known as *Haezendonck-Goovaerts risk measures*; see Haezendonck and Goovaerts (34), Delbaen (18), Goovaerts, Kaas, Dhaene and Tang (33) and Bellini and Rosazza Gianin (3; 4) for the classical (non-robust) definitions. We explicated that Orlicz premia can be interpreted to assess the stochastic nature of returns—they are return risk measures—, in contrast to the common use of monetary risk measures to assess the stochastic nature of a position's monetary value. The class of return risk measures encompasses interesting subclasses of risk measures, such as p-norms, that are not included in the class of monetary measures of risk.

In this paper we extend to a dynamic setting the static robust return risk measures introduced in Bellini, Laeven and Rosazza Gianin (5). We extensively analyze the properties of the resulting dynamic robust return risk measures. Furthermore, we provide characterization results of their time-consistency. We show in particular that the only time-consistent dynamic Orlicz premia are conditional p-norms. We also show that time-consistency of dynamic robust Orlicz premia and of the associated Haezendonck-Goovaerts risk measures are intimately related.

The remainder of this paper is organized as follows: In Section 2 we introduce our notation and setting and recall some preliminaries. In Section 3 we introduce dynamic Orlicz premia and analyze their properties. In Section

4 we consider their robust extension. In Section 5 we introduce dynamic robust Haezendonck-Goovaerts risk measures. In Section 6 we analyze time-consistency properties.

2. Preliminaries and basic definitions

Throughout the paper, we work on a nonatomic probability space $(\Omega, \mathcal{F}, P)$. All equalities and inequalities between random variables are meant to hold P-a.s. without further specification. We denote by $L^\infty(P)$, $L^\infty_+(\mathcal{F}, P)$, and $L^\infty_{++}(P)$ the sets of P-a.s. bounded, P-a.s. bounded non-negative, and P-a.s. bounded strictly positive random variables, respectively. We assume that positive realizations of random variables represent losses. A risk measure $\rho\colon L^\infty(P) \to \mathbb{R}$ is said to be:

- monotone, if $X \leq Y \Rightarrow \rho(X) \leq \rho(Y)$
- strictly monotone, if $X \leq Y$ and $P(X < Y) > 0 \Rightarrow \rho(X) < \rho(Y)$
- translation invariant, if $\rho(X + h) = \rho(X) + h$, $\forall h \in \mathbb{R}, \forall X \in L^\infty$
- monetary, if it is monotone, translation invariant and satisfies $\rho(0) = 0$
- convex, if $\rho(\alpha X + (1-\alpha)Y) \leq \alpha\rho(X) + (1-\alpha)\rho(Y)$, $\forall \alpha \in [0, 1], \forall X, Y \in L^\infty$
- positively homogeneous, if $\rho(\lambda X) = \lambda\rho(X)$, $\forall \lambda \geq 0, \forall X \in L^\infty$
- subadditive, if $\rho(X + Y) \leq \rho(X) + \rho(Y), \forall X, Y \in L^\infty$
- coherent, if it is monotone, translation invariant, positively homogeneous and subadditive
- law-invariant, if $X \overset{d}{=} Y \Rightarrow \rho(X) = \rho(Y)$.

A risk measure ρ has the Fatou property if

$$X_n \overset{P}{\to} X,\ \|X_n\|_\infty \leq k \Rightarrow \rho(X) \leq \liminf_{n\to+\infty} \rho(X_n),$$

while it has the stronger Lebesgue property if

$$X_n \overset{P}{\to} X,\ \|X_n\|_\infty \leq k \Rightarrow \rho(X) = \lim_{n\to+\infty} \rho(X_n).$$

With our sign conventions, for monotone and convex risk measures the Lebesgue property is equivalent to continuity from above, i.e.,

$$X_n \downarrow X \Rightarrow \rho(X_n) \to \rho(X),$$

while the Fatou property is equivalent to continuity from below, that is,

$$X_n \uparrow X \Rightarrow \rho(X_n) \to \rho(X).$$

See Delbaen (18) and Föllmer and Schied (27).

We let $\Phi\colon [0,+\infty) \to [0,+\infty)$ be strictly increasing and convex, with $\Phi(0) = 0$, $\Phi(1) = 1$, and $\Phi(+\infty) = +\infty$. Such a Φ is referred to as a Young function.

Definition 1. *Let Φ be a Young function. For a random loss $X \in L^\infty_+(\Omega, \mathcal{F}, P)$, the Orlicz premium is defined by*

$$H^\Phi(X) := \inf \left\{ k > 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] \leq 1 \right\}.$$

One easily verifies that Orlicz premia are strictly monotone, positively homogeneous, subadditive, law invariant, have the Lebesgue property, and satisfy $H^\Phi(c) = c$, for every $c \geq 0$. For $\Phi(x) = x^p$, with $p \geq 1$, clearly $H^\Phi(X) = \|X\|_p$. Orlicz premia are law-invariant norms and their natural domain is the nonnegative cone of an Orlicz space

$$L^\Phi_+ := \left\{ X \geq 0 \;\middle|\; \mathbb{E}\left[\Phi\left(\frac{X}{k}\right)\right] < +\infty, \text{ for some } k > 0 \right\}.$$

We refer to Rao and Ren (44), Haezendonck and Goovaerts (34), Bellini and Rosazza Gianin (3; 4) and Cheridito and Li (13; 14) for further properties of Orlicz premia and Orlicz spaces. Notice that if $X \in L^\infty_+$, $X \neq 0$, then

$$\mathbb{E}_P\left[\Phi\left(\frac{X}{H^\Phi(X)}\right)\right] = 1,$$

and moreover $H^\Phi(X) = 1 \iff \mathbb{E}[\Phi(X)] = 1$.

3. Dynamic Orlicz premia

In this section we extend the definition of Orlicz premia to a dynamic setting. On the probability space $(\Omega, \mathcal{F}, P)$ we fix a filtration $(\mathcal{F}_t)_{t\in[0,T]}$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. For each $t \in (0, T]$, we assume that $(\Omega, \mathcal{F}_t, P)$ is nonatomic.

Definition 2. *Let Φ be a Young function and let $t, u \in [0, T]$ with $t \leq u$. For $X \in L^\infty_+(\mathcal{F}_u)$, the dynamic Orlicz premium $H_t^\Phi \colon L^\infty_+(\mathcal{F}_u) \to L^\infty_+(\mathcal{F}_t)$ is*

$$H_t^\Phi(X) := \operatorname{ess\,inf}\left\{ h_t \in L^\infty_{++}(\mathcal{F}_t) \;\middle|\; \mathbb{E}_P\left[\Phi\left(\frac{X}{h_t}\right) \;\middle|\; \mathcal{F}_t\right] \leq 1 \right\}. \tag{1}$$

Recall that the essential infimum of a family of $\mathcal{F}_t$-measurable functions $\{h_\alpha\}_{\alpha\in I}$ is the P-a.s. unique $\mathcal{F}_t$-measurable function Z such that $Z \geq h_\alpha$ for each $\alpha \in I$, and if Z' is another $\mathcal{F}_t$-measurable function satisfying $Z' \geq h_\alpha$ for each $\alpha \in I$ then $Z' \geq Z$ (see e.g., Föllmer and Schied (27)). Since $X \in L^\infty_+(\mathcal{F}_u)$, the set

$$\left\{ h_t \in L^\infty_{++}(\mathcal{F}_t) \;\middle|\; \mathbb{E}_P\left[\Phi\left(\frac{X}{h_t}\right) \middle| \mathcal{F}_t\right] \leq 1 \right\}$$

is non void, so the definition is well-posed. Clearly, $H_0^\Phi = H^\Phi$ and $H_t^\Phi(0) = 0$. We will also consider conditional Orlicz premia denoted for a general σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ by

$$H_{\mathcal{G}}^\Phi(X) := \operatorname{ess\,inf}\left\{ h_t \in L^\infty_{++}(\mathcal{G}) \;\middle|\; \mathbb{E}_P\left[\Phi\left(\frac{X}{h_t}\right) \;\middle|\; \mathcal{G}\right] \leq 1 \right\}. \tag{2}$$

The properties of H_t^Φ are similar to those of H^Φ and are reported in the following proposition.

Proposition 3. *Let Φ be a Young function, let $s, t \in [0, T]$ with $s \leq t$ and let $H_t^\Phi \colon L^\infty_+(\mathcal{F}_T) \to L^\infty_+(\mathcal{F}_t)$ be defined as in (1). Then for each $X, Y \in L^\infty_+(\mathcal{F}_T)$ it holds that:*

(a) $X \leq Y \Rightarrow H_t^\Phi(X) \leq H_t^\Phi(Y)$

(a') *Let $X, Y > 0$. If $X \leq Y, P(X < Y) > 0 \Rightarrow H_t^\Phi(X) \leq H_t^\Phi(Y)$ with $P(H_t^\Phi(X) < H_t^\Phi(Y)) > 0$*

(b) $H_t^\Phi(X + Y) \leq H_t^\Phi(X) + H_t^\Phi(Y)$

(c) $X \in L^\infty_+(\mathcal{F}_t) \Rightarrow H^\Phi_t(X) = X$

(d) $H^\Phi_t(\lambda_s X) = \lambda_s H^\Phi_t(X)$, $\forall \lambda_s \in L^\infty_{++}(\mathcal{F}_s)$

(e) $H^\Phi_t(X + \eta_s) \le H^\Phi_t(X) + \eta_s$, $\forall \eta_s \in L^\infty_+(\mathcal{F}_s)$

(f) $\mathbb{E}_P[X|\mathcal{F}_t] \le H^\Phi_t(X) \le \|X\|_\infty$

(g) $A \in \mathcal{F}_t \Rightarrow H^\Phi_t(X 1_A) = 1_A H^\Phi_t(X)$

(h) *if* $X > 0$, *then*

$$\mathbb{E}_P\left[\Phi\left(\frac{X}{H^\Phi_t(X)}\right)\middle|\mathcal{F}_t\right] = 1$$

(i) $H^\Phi_t(X) = 1 \iff \mathbb{E}[\Phi(X)|\mathcal{F}_t] = 1$

(j) *if* $X_n \downarrow X$, *or if* $X_n \uparrow X$, *or if* $X_n \to X$ *with* $\|X_n\| \le k$, *then*

$$H^\Phi_t(X_n) \to H^\Phi_t(X).$$

(k) *if* F_t *is a regular version of the conditional distribution of* X *given* $\mathcal{F}_t$, *then*

$$H^\Phi_t(X) = H^\Phi\left(F_t(\cdot, \omega)\right)$$

(l) *if* X *is independent of* $\mathcal{F}_t$, *then* $H^\Phi_t(X) = H^\Phi(X)$.

Proof. The proof of (a) and (b) is straightforward and similar to the static case. (a') Let $X, Y > 0$, $X \le Y$ with $P(X < Y) > 0$. By item (h),

$$\begin{aligned}
1 &= \mathbb{E}_P\left[\Phi\left(\frac{Y}{H^\Phi_t(Y)}\right)\ \middle|\ \mathcal{F}_t\right] \\
&= \mathbb{E}_P\left[\Phi\left(\frac{Y}{H^\Phi_t(Y)}\right) 1_{\{X=Y\}} + \Phi\left(\frac{Y}{H^\Phi_t(Y)}\right) 1_{\{X<Y\}}\ \middle|\ \mathcal{F}_t\right] \\
&> \mathbb{E}_P\left[\Phi\left(\frac{X}{H^\Phi_t(Y)}\right)\ \middle|\ \mathcal{F}_t\right].
\end{aligned}$$

Since $\Phi\left(X/h\right)$ is strictly decreasing in h, the thesis follows.
(c) If $X \in L^{\infty}_{+}(\mathcal{F}_t)$ then from the properties of Φ it follows that

$$\begin{aligned}
&\operatorname{ess\,inf}\left\{h_t \in L^{\infty}_{++}(\mathcal{F}_t) \;\middle|\; \mathbb{E}_P\left[\Phi\left(\frac{X}{h_t}\right) \;\middle|\; \mathcal{F}_t\right] \leq 1\right\} \\
&= \operatorname{ess\,inf}\left\{h_t \in L^{\infty}_{++}(\mathcal{F}_t) \;\middle|\; \Phi\left(\frac{X}{h_t}\right) \leq 1\right\} \\
&= \operatorname{ess\,inf}\left\{h_t \in L^{\infty}_{++}(\mathcal{F}_t) \;\middle|\; X \leq h_t\right\} = X.
\end{aligned}$$

(d) It holds that

$$\begin{aligned}
&\operatorname{ess\,inf}\left\{h_t \in L^{\infty}_{++}(\mathcal{F}_t) \;\middle|\; \mathbb{E}_P\left[\Phi\left(\frac{\lambda_s X}{h_t}\right) \middle|\mathcal{F}_t\right] \leq 1\right\} \\
&= \operatorname{ess\,inf}\left\{\lambda_s h_t \in L^{\infty}_{++}(\mathcal{F}_t) \;\middle|\; \mathbb{E}_P\left[\Phi\left(\frac{X}{h_t}\right) \middle|\mathcal{F}_t\right] \leq 1\right\} \\
&= \lambda_s H^{\Phi}_t(X).
\end{aligned}$$

(g) follows immediately from (d).
(e) follows from (b) and (c).
(f) If $\mathbb{E}[X|\mathcal{F}_t] > 0$, then from the conditional Jensen inequality it follows that

$$\mathbb{E}\left[\Phi\left(\frac{X}{\mathbb{E}[X|\mathcal{F}_t]}\right)\middle|\,\mathcal{F}_t\right] \geq \Phi(1) = 1,$$

which gives the first part of the thesis. If instead $\mathbb{E}[X|\mathcal{F}_t] = 0$ on $A \in \mathcal{F}_t$ with $P(A) > 0$, then

$$\mathbb{E}[X|\mathcal{F}_t] = \mathbb{E}[X1_A|\mathcal{F}_t] + \mathbb{E}[X1_{A^c}|\mathcal{F}_t] \leq H^{\Phi}_t(X1_A) + H^{\Phi}_t(X1_{A^c}) = H^{\Phi}_t(X),$$

where the last equality follows from (g). The second part follows from (a).
(h) The set

$$A_t := \left\{h_t \in L^{\infty}_{++} : E_P\left[\Phi\left(\frac{X}{h_t}\right)\middle|\mathcal{F}_t\right] \leq 1\right\}$$

is downward directed, hence there exists $z_n \in A_t$, $z_n \downarrow H^{\Phi}_t(X)$, P-a.s. From the dominated convergence theorem, it follows that

$$E_P\left[\Phi\left(\frac{X}{H^{\Phi}_t(X)}\right)\middle|\mathcal{F}_t\right] \leq 1.$$

Let now $\bar{z}_n \in L^\infty_{++}$, with $\bar{z}_n < H^\Phi_t(X)$ and $\bar{z}_n \uparrow H^\Phi_t(X)$. Since $H^\Phi_t(X) = \operatorname{ess\,inf} A_t$, it holds that

$$E_P\left[\Phi\left(\frac{X}{\bar{z}_n}\right)\Big|\mathcal{F}_t\right] > 1,$$

and again from the dominated convergence theorem,

$$E_P\left[\Phi\left(\frac{X}{H^\Phi_t(X)}\right)\Big|\mathcal{F}_t\right] \geq 1,$$

which gives the thesis.
(i) follows immediately from (h).
(j) Let $X_n \downarrow X$. Then by monotonicity $H_n := H^\Phi_t(X_n) \downarrow H \geq H^\Phi_t(X)$.
If $H > 0$, then it follows that

$$\mathbb{E}_P\left[\Phi\left(\frac{X_n}{H}\right)\Big|\mathcal{F}_t\right] \geq 1,$$

and by the dominated convergence theorem

$$\mathbb{E}_P\left[\Phi\left(\frac{X}{H}\right)\Big|\mathcal{F}_t\right] \geq 1.$$

Similarly,

$$\mathbb{E}_P\left[\Phi\left(\frac{X}{H_n}\right)\Big|\mathcal{F}_t\right] \leq 1,$$

and by the dominated convergence theorem

$$\mathbb{E}_P\left[\Phi\left(\frac{X}{H}\right)\Big|\mathcal{F}_t\right] \leq 1,$$

so

$$\mathbb{E}_P\left[\Phi\left(\frac{X}{H}\right)\Big|\mathcal{F}_t\right] = 1,$$

which from (h) implies that $H = H^\Phi_t(X)$. If instead $H = 0$ on $A \in \mathcal{F}_t$, then also $H^\Phi_t(X) = 0$ on A, and the thesis follows by (g). The proof of continuity from below is similar. To prove the last part of the thesis, let

$$Z_n = \sup_{k \geq n} X_k,\ Y_n = \inf_{k \geq n} X_k.$$

Then $Z_n \geq X_n$, $Z_n \downarrow X$ and $Y_n \leq X_n$, $Y_n \uparrow X$, so from monotonicity and the first part of the thesis it follows that $H_t^\Phi(X_n) \to H_t^\Phi(X)$.

(k) Let F_t be a regular version of the conditional distribution of X given $\mathcal{F}_t$, that is let $F_t \colon \mathbb{R} \times \Omega \to [0,1]$ be such that for each $\omega \in \Omega$, $F_t(\cdot, \omega)$ is a distribution function on $\mathbb{R}$ and for each $x \in \mathbb{R}$ it holds $F_t(x, \cdot) = \mathbb{E}_P[1_{\{X \leq x\}} | \mathcal{F}_t]$. Since

$$\mathbb{E}_P \left[\Phi \left(\frac{X}{h_t} \right) \middle| \mathcal{F}_t \right] = \int \Phi \left(x/h \right) dF_t(x, \omega) \ P\text{-a.s.},$$

the thesis follows.

(l) if X is independent of $\mathcal{F}_t$, then $F_t(x, \omega) = F(x) = P(X \leq x)$ is a regular version of the conditional distribution of X given $\mathcal{F}_t$ that does not depend on ω, so the thesis follows from (k). □

Example 4. *If $\Phi(x) = x^p$ with $p \geq 1$, then*

$$H_t^\Phi(X) = \left(\mathbb{E}[X^p | \mathcal{F}_t] \right)^{1/p}.$$

4. Dynamic robust Orlicz premia

In Bellini et al. (5) we introduced robust Orlicz premia, arising from a penalized worst-case approach under ambiguity with respect to the true measure P. We considered two canonical cases of ambiguity averse preferences: variational preferences as in Maccheroni, Marinacci and Rustichini (41) and homothetic preferences as in Chateauneuf and Faro (10). We recall the basic definitions and notations. We denote by $\mathcal{Q}$ the set of all probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to P.

Definition 5. *Let Φ be a Young function, let $c \colon \mathcal{Q} \to [0, +\infty]$ be a penalty function satisfying $\inf_{Q \in \mathcal{Q}} c(Q) = 0$, and let $\beta \colon \mathcal{Q} \to [0,1]$ be a confidence function satisfying $\sup_{Q \in \mathcal{Q}} \beta(Q) = 1$. The robust Orlicz premia are defined by*

$$H^{\Phi,c}(X) := \inf \left\{ k > 0 \;\middle|\; \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[\Phi \left(\frac{X}{k} \right) \right] - c(Q) \leq 1 \right\}.$$

$$H^{\Phi,\beta}(X) := \inf \left\{ k > 0 \;\middle|\; \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[\beta(Q) \Phi \left(\frac{X}{k} \right) \right] \leq 1 \right\}.$$

The corresponding dynamic robust Orlicz premia are defined as follows:

Definition 6. *Let Φ be a Young function, let $\mathcal{C} = \{c_t\}_{t\in[0,T]}$ be a family of $\mathcal{F}_t$-measurable penalty functions $c_t \colon \mathcal{Q} \to L^0_+(\mathcal{F}_t)$ satisfying $\inf_{Q\in\mathcal{Q}} c_t(Q) = 0$, and let $\mathcal{B} = \{\beta_t\}_{t\in[0,T]}$ be a family of $\mathcal{F}_t$-measurable confidence functions $\beta_t \colon \mathcal{Q} \to L^0_{[0,1]}(\mathcal{F}_t)$ satisfying $\sup_{Q\in\mathcal{Q}} \beta_t(Q) = 1$, for each $t \in [0,T]$. Let $t, u \in [0,T]$ with $t \leq u$. For $X \in L^\infty_+(\mathcal{F}_u)$, we define*

$$H_t^{\Phi,\mathcal{C}}(X) := \operatorname{ess\,inf}\left\{ h_t \in L^\infty_{++}(\mathcal{F}_t) \;\middle|\; \operatorname*{ess\,sup}_{Q\in\mathcal{Q}} \left\{ \mathbb{E}_Q\left[\Phi\left(\frac{X}{h_t}\right) \middle| \mathcal{F}_t \right] - c_t(Q) \right\} \leq 1 \right\} \tag{3}$$

$$H_t^{\Phi,\mathcal{B}}(X) := \operatorname{ess\,inf}\left\{ h_t \in L^\infty_{++}(\mathcal{F}_t) \;\middle|\; \operatorname*{ess\,sup}_{Q\in\mathcal{Q}} \left\{ \mathbb{E}_Q\left[\beta_t(Q)\, \Phi\left(\frac{X}{h_t}\right) \middle| \mathcal{F}_t \right] \right\} \leq 1 \right\} \tag{4}$$

Notice however that if, in (3), we define

$$\rho_t(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}} \left\{ \mathbb{E}_Q\left[X \mid \mathcal{F}_t \right] - c_t(Q) \right\}, \tag{5}$$

or similarly if, in (4), we define

$$\rho_t(X) := \operatorname*{ess\,sup}_{Q\in\mathcal{Q}} \left\{ \beta_t(Q)\, \mathbb{E}_Q\left[X \mid \mathcal{F}_t \right] \right\}, \tag{6}$$

then ρ_t is in both cases a dynamic risk measure that satisfies monotonicity and convexity, and (3) and (4) can be rewritten in a unified way as follows:

$$H_t^{\Phi,\rho}(X) := \operatorname{ess\,inf}\left\{ h_t \in L^\infty_{++}(\mathcal{F}_t) \;\middle|\; \rho_t\left(\Phi\left(\frac{X}{h_t}\right)\right) \leq 1 \right\}. \tag{7}$$

In other words, dynamic robust Orlicz premia arise by replacing the conditional expectation operator in (1) with a more general dynamic risk measure ρ_t that is convex. We begin by illustrating some simple special cases of Definition 6 in the case of variational preferences.

Example 7. *Let $c_t = 0$ for each $t \in [0,T]$. Then, $\rho_t^{\mathcal{C}}(X) = \operatorname{ess\,sup}[X \mid \mathcal{F}_t]$, and*

$$\begin{aligned} H_t^{\Phi,\mathcal{C}}(X) &= \operatorname{ess\,inf}\left\{ h_t \in L^\infty_{++}(\mathcal{F}_t) \;\middle|\; \operatorname{ess\,sup}\left[\Phi\left(\frac{X}{h_t}\right) \;\middle|\; \mathcal{F}_t \right] \leq 1 \right\} \\ &= \operatorname{ess\,sup}[X \mid \mathcal{F}_t]. \end{aligned}$$

Example 8. *Let $\Phi(x) = x^p$, with $p \geq 1$, and let*

$$c_t(Q) = \begin{cases} 0 & \text{if } Q \in \mathcal{S}_t \\ +\infty & \text{if } Q \notin \mathcal{S}_t, \end{cases}$$

with $\mathcal{S}_t \subset \mathcal{Q}$, for $t \in [0, T]$. Then,

$$H_t^{\Phi,\mathcal{C}}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{S}_t} \left(\mathbb{E}_Q\left[X^p \middle| \mathcal{F}_t\right]\right)^{1/p}.$$

In particular, for $p = 1$ we get

$$H_t^{\Phi,\mathcal{C}}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{S}_t} \mathbb{E}_Q\left[X \middle| \mathcal{F}_t\right],$$

which is a dynamic coherent risk measure in the usual sense.

Example 9. *Let $\Phi(x) = x^p$, with a general $\mathcal{C}$. Then,*

$$H_t^{\Phi,\mathcal{C}}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{Q}} \left(\frac{\mathbb{E}_Q\left[X^p \middle| \mathcal{F}_t\right]}{1 + c_t(Q)}\right)^{1/p}.$$

Indeed, Definition 6 becomes

$$H_t^{\Phi,\mathcal{C}}(X) = \operatorname*{ess\,inf}\left\{h_t \in L^\infty_{++}(\mathcal{F}_t) \;\middle|\; \operatorname*{ess\,sup}_{Q \in \mathcal{Q}} \left\{\frac{\mathbb{E}_Q\left[X^p \middle| \mathcal{F}_t\right]}{h_t^p} - c_t(Q)\right\} \leq 1\right\},$$

and the condition

$$\operatorname*{ess\,sup}_{Q \in \mathcal{Q}} \left\{\frac{\mathbb{E}_Q\left[X^p \middle| \mathcal{F}_t\right]}{h_t^p} - c_t(Q)\right\} \leq 1$$

is equivalent to

$$h_t \geq \operatorname*{ess\,sup}_{Q \in \mathcal{Q}} \left(\frac{\mathbb{E}_Q\left[X^p \middle| \mathcal{F}_t\right]}{1 + c_t(Q)}\right)^{1/p}.$$

Notice that the quantity $1/(1 + c_t(Q))$ is an $\mathcal{F}_t$-measurable random variable taking values in $[0, 1]$, so it may be interpreted as a discount factor.

Some of the properties of dynamic Orlicz premia remain valid also in the robust case; we list them in the following proposition.

Proposition 10. *Let Φ be a Young function, let $s, t \in [0, T]$ with $s \leq t$, and let $H_t^{\Phi,\rho}\colon L_+^\infty(\mathcal{F}_T) \to L_+^\infty(\mathcal{F}_t)$ be as in (7). Then, for each $X, Y \in L_+^\infty(\mathcal{F}_T)$, it holds that:*

(a) $X \leq Y \Rightarrow H_t^{\Phi,\rho}(X) \leq H_t^{\Phi,\rho}(Y)$

(b) $H_t^{\Phi,\rho}(X+Y) \leq H_t^{\Phi,\rho}(X) + H_t^{\Phi,\rho}(Y)$

(c) $X \in L_+^\infty(\mathcal{F}_t) \Rightarrow H_t^{\Phi,\rho}(X) = X$

(d) $H_t^{\Phi,\rho}(\lambda_s X) = \lambda_s H_t^{\Phi,\rho}(X)$, $\forall \lambda_s \in L_{++}^\infty(\mathcal{F}_s)$

(e) $H_t^{\Phi,\rho}(X + \eta_s) \leq H_t^{\Phi,\rho}(X) + \eta_s$, $\forall \eta_s \in L_+^\infty(\mathcal{F}_s)$

(f) $H_t^{\Phi,\rho}(X) \leq \|X\|_\infty$

(g) $A \in \mathcal{F}_t \Rightarrow H_t^{\Phi,\rho}(X 1_A) = 1_A\, H_t^{\Phi,\rho}(X)$

(h) if ρ_t has the Lebesgue property, then for $X > 0$ it holds that

$$\rho_t \left(\Phi \left(\frac{X}{H_t^{\Phi,\rho}(X)} \right) \right) = 1$$

(i) if ρ_t has the Lebesgue property, then $H_t^{\Phi,\rho}(X) = 1 \iff \rho_t(\Phi(X)) = 1$

(j) if ρ_t has the Lebesgue property, then if $X_n \downarrow X$, or if $X_n \uparrow X$, or if $X_n \to X$ with $\|X_n\| \leq k$, it follows that

$$H_t^{\Phi,\rho}(X_n) \to H_t^{\Phi,\rho}(X)$$

(k) if ρ_t is conditionally law-invariant, then also $H_t^{\Phi,\rho}$ is conditionally law-invariant.

Proof. (a) follows immediately from the monotonicity of ρ_t. (b) Notice that

$$\begin{aligned}
&\rho_t \left(\Phi \left(\frac{X+Y}{H_t^{\Phi,\rho}(X) + H_t^{\Phi,\rho}(Y)} \right) \right) \\
&\leq \rho_t \left(\frac{H_t^{\Phi,\rho}(X)}{H_t^{\Phi,\rho}(X) + H_t^{\Phi,\rho}(Y)} \Phi \left(\frac{X}{H_t^{\Phi,\rho}(X)} \right) + \frac{H_t^{\Phi,\rho}(Y)}{H_t^{\Phi,\rho}(X) + H_t^{\Phi,\rho}(Y)} \Phi \left(\frac{Y}{H_t^{\Phi,\rho}(Y)} \right) \right) \\
&\leq \frac{H_t^{\Phi,\rho}(X)}{H_t^{\Phi,\rho}(X) + H_t^{\Phi,\rho}(Y)} \rho_t \left(\Phi \left(\frac{X}{H_t^{\Phi,\rho}(X)} \right) \right) + \frac{H_t^{\Phi,\rho}(Y)}{H_t^{\Phi,\rho}(X) + H_t^{\Phi,\rho}(Y)} \rho_t \left(\Phi \left(\frac{Y}{H_t^{\Phi,\rho}(Y)} \right) \right) \\
&\leq 1,
\end{aligned}$$

where the first inequality follows from the convexity of Φ and the second from the convexity of ρ_t, from which the thesis follows.
(c) Notice that if $X \in L_+^\infty(\mathcal{F}_t)$, then $\rho_t(X) = X$, as a consequence of the hypothesis $\inf_{Q\in Q} c_t(Q) = 0$ and $\sup_{Q\in Q} \beta_t(Q) = 1$. The thesis then follows as in item (c) of Proposition 3. The proofs of (d), (e), (f), (g), (h), (i), (j), (k) are identical to the corresponding items in Proposition 3, since under the assumption that ρ_t has the Lebesgue property, the dominated convergence theorem can still be applied. □

Summing up, the relevant properties of ρ_t in Proposition 10 are monotonicity (for (a)), convexity (for (b)), constancy (for (c)), the Lebesgue property (for (h), (i) and (j)), and conditional law-invariance (for (k)). In particular, neither translation invariance of ρ_t (which is satisfied in the case of variational preferences (5) but not in the case of homothetic preferences (6)), nor conditional positive homogeneity (which is satisfied in the case of homothetic preferences (6) but not in the case of variational preferences (5)) play a role in the proof of Proposition 10. Sufficient conditions on the penalty functions in (5) that guarantee the validity of these properties are well known in the literature on dynamic convex risk measures; we refer e.g., to Detlefsen and Scandolo (22), Föllmer and Schied (27). The case of homothetic preferences is less explored.

5. Dynamic robust Haezendonck-Goovaerts risk measures

Dynamic robust Orlicz premia do not satisfy a translation invariance property, and are defined only for nonnegative losses. A construction via optimized translation-invariant extensions (Rockafellar and Uryasev (46) and Rockafellar, Uryasev and Zabarankin (47)) that resolves both issues has been suggested in Goovaerts et al. (33) (see also Bellini and Rosazza Gianin (3; 4)), leading to the so-called Haezendonck-Goovaerts risk measures (HG henceforth), of which a robust version has been introduced in Bellini et al. (5). Their dynamic extension can be given as follows.

Definition 11. *Let Φ be a Young function, let $s, t \in [0,T]$ with $s \le t$, and let $H_t^{\Phi,\rho}$ be as in (7). For $X \in L^\infty(\mathcal{F}_T)$, we define*

$$HG_t^{\Phi,\rho}(X) := \operatorname*{ess\,inf}_{x_t \in L^\infty(\mathcal{F}_t)} \{x_t + H_t^{\Phi,\rho}\left((X - x_t)^+\right)\}. \tag{8}$$

All the properties of $H_t^{\Phi,\rho}$ are inherited also by $HG_t^{\Phi,\rho}$.

Proposition 12. *Let Φ be a Young function, let $s, t \in [0,T]$ with $s \leq t$, and let $H_t^{\Phi,\rho}$ be as in (7). Then, for each $X, Y \in L^\infty(\mathcal{F}_T)$, the following hold:*

(a) $X \leq Y \Rightarrow HG_t^{\Phi,\rho}(X) \leq HG_t^{\Phi,\rho}(Y)$

(b) $HG_t^{\Phi,\rho}(X+Y) \leq HG_t^{\Phi,\rho}(X) + HG_t^{\Phi,\rho}(Y)$

(c) $X \in L_+^\infty(\mathcal{F}_t) \Rightarrow HG_t^{\Phi,\rho}(X) = X$

(d) $HG_t^{\Phi,\rho}(\lambda_s X) = \lambda_s HG_t^{\Phi,\rho}(X)$, $\forall \lambda_s \in L_{++}^\infty(\mathcal{F}_s)$

(e) $HG_t^{\Phi,\rho}(X+\eta_t) = HG_t^{\Phi,\rho}(X) + \eta_t$, $\forall \eta_t \in L^\infty(\mathcal{F}_t)$

(f) $HG_t^{\Phi,\rho}(X) \leq \|X\|_\infty$

(g) $A \in \mathcal{F}_t \Rightarrow HG_t^{\Phi,\rho}(X 1_A) = 1_A\, HG_t^{\Phi,\rho}(X)$

(h) if ρ_t satisfies the Lebesgue property, then

$$X_n \downarrow X \Rightarrow HG_t^{\Phi,\mathcal{C}}(X_n) \to HG_t^{\Phi,\mathcal{C}}(X), \; P\text{-a.s.}$$

(i) if ρ_t is conditionally law-invariant, then also $HG_t^{\Phi,\rho}$ is conditionally law-invariant.

Proof. (a) follows immediately from the monotonicity of $H_t^{\Phi,\rho}$.

(b) For any $X, Y \in L^\infty(\mathcal{F}_T)$ it holds that

$$\begin{aligned}
&HG_t^{\Phi,\rho}(X+Y) = \operatorname*{ess\,inf}_{x_t^X, x_t^Y \in L^\infty(\mathcal{F}_t)} \{x_t^X + x_t^Y + H_t^{\Phi,\rho}\left((X+Y-x_t^X-x_t^Y)^+\right)\} \\
&\leq \operatorname*{ess\,inf}_{x_t^X, x_t^Y \in L^\infty(\mathcal{F}_t)} \{x_t^X + x_t^Y + H_t^{\Phi,\rho}\left((X-x_t^X)^+\right) + H_t^{\Phi,\rho}\left((Y-x_t^Y)^+\right)\} \\
&= \operatorname*{ess\,inf}_{x_t^X \in L^\infty(\mathcal{F}_t)} \{x_t^X + H_t^{\Phi,\rho}\left((X-x_t^X)^+\right)\} + \operatorname*{ess\,inf}_{x_t^Y \in L^\infty(\mathcal{F}_t)} \{x_t^Y + H_t^{\Phi,\rho}\left((Y-x_t^Y)^+\right)\} \\
&= HG_t^{\Phi,\rho}(X) + HG_t^{\Phi,\rho}(Y),
\end{aligned}$$

where the first inequality follows from the subadditivity of the positive part and of $H_t^{\Phi,\rho}$.

(c) If $X \in L_+^\infty(\mathcal{F}_t)$, then

$$HG_t^{\Phi,\rho}(X) = \operatorname*{ess\,inf}_{x_t \in L^\infty(\mathcal{F}_t)} \{x_t + H_t^{\Phi,\rho}\left((X-x_t)^+\right)\} = X.$$

(d) If $\lambda_s \in L^\infty_{++}(\mathcal{F}_s)$, then

$$\begin{aligned} HG_t^{\Phi,\rho}(\lambda_s X) &= \operatorname*{ess\,inf}_{x_t \in L^\infty(\mathcal{F}_t)} \{x_t + H_t^{\Phi,\rho}\left((\lambda_s X - x_t)^+\right)\} \\ &= \operatorname*{ess\,inf}_{x_t \in L^\infty(\mathcal{F}_t)} \{\lambda_s x_t + H_t^{\Phi,\rho}\left((\lambda_s X - \lambda_s x_t)^+\right)\} = \lambda_s HG_t^{\Phi,\rho}(X), \end{aligned}$$

from the conditional positive homogeneity of $H_t^{\Phi,\rho}$. (e) follows immediately from (8). (f) and (g) can be proved as in Proposition 10. (h) from Proposition 10, item (j), it follows that $H_t^{\Phi,\rho}$ is continuous from above, so, if $X_n \downarrow X$, then

$$\begin{aligned} \inf_n HG_t^{\Phi,\rho}(X_n) &= \inf_n \operatorname*{ess\,inf}_{x_t \in L^\infty(\mathcal{F}_t)} \{x_t + H_t^{\Phi,\rho}\left((X_n - x_t)^+\right)\} \\ &= \operatorname*{ess\,inf}_{x_t \in L^\infty(\mathcal{F}_t)} \inf_n \{x_t + H_t^{\Phi,\rho}\left((X_n - x_t)^+\right)\} \\ &= \operatorname*{ess\,inf}_{x_t \in L^\infty(\mathcal{F}_t)} \{x_t + H_t^{\Phi,\rho}\left((X - x_t)^+\right)\} = HG_t^{\Phi,\rho}(X), \end{aligned}$$

from which the thesis follows. (i) can be proved as in the static case. □

The preceding proposition shows that dynamic robust HG risk measures are dynamic coherent risk measures in the sense of Delbaen (19) and Riedel (45), so they possess a dual representation in terms of essential suprema of conditional expectations, which is given in the following proposition. We denote by $\mathcal{Q}_t \subset \mathcal{Q}$ the set of probability measures on $(\Omega, \mathcal{F})$ such that $Q = P$ on $\mathcal{F}_t$.

Proposition 13. *Let Φ be a Young function, let $t, u \in [0,T]$ with $t \le u$, and let $HG_t^{\Phi,\rho}$ be as in Definition 11, with ρ_t being continuous from above. Then, for each $X \in L^\infty(\mathcal{F}_u)$,*

$$HG_t^{\Phi,\rho}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{R}_t} \mathbb{E}_Q\left[X \,|\, \mathcal{F}_t\right], \tag{9}$$

where

$$\mathcal{R}_t := \{Q \in \mathcal{Q}_t \,|\, \mathbb{E}_Q\left[Z \,|\, \mathcal{F}_t\right] \le H_t^{\Phi,\rho}(Z^+), \text{ for each } Z \in L^\infty(\mathcal{F}_t)\}.$$

Proof. By Detlefsen and Scandolo (22) (see also Delbaen (19) and Klöppel and Schweizer (35)) it follows that

$$HG_t^{\Phi,\rho}(X) = \operatorname*{ess\,sup}_{Q \in \mathcal{R}_t} \mathbb{E}_Q\left[X \,|\, \mathcal{F}_t\right],$$

where $\mathcal{R}_t = \{Q \in \mathcal{Q}_t \mid \mathbb{E}_Q[Y|\mathcal{F}_t] \le HG_t^{\Phi,\rho}(Y),\text{ for each } Y \in L^\infty(\mathcal{F}_u)\}$. Furthermore, from (8) it follows that

$$\mathbb{E}_Q[Y|\mathcal{F}_t] \le HG_t^{\Phi,\rho}(Y), \quad \forall Y \in L^\infty(\mathcal{F}_u),$$

is equivalent to

$$\mathbb{E}_Q[Y - x_t|\mathcal{F}_t] \le H_t^{\Phi,\rho}\left((Y - x_t)^+\right), \quad \forall x_t \in L^\infty(\mathcal{F}_t), Y \in L^\infty(\mathcal{F}_u),$$

and, moreover, to

$$\mathbb{E}_Q[Z|\mathcal{F}_t] \le H_t^{\Phi,\rho}(Z^+), \quad \forall Z \in L^\infty(\mathcal{F}_u),$$

from which the thesis follows. □

6. Time-consistency properties

We start by recalling several definitions related to time-consistency that have been considered in the literature.

Definition 14. *For $t \in [0,T]$, let $\pi_t \colon L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ be a dynamic risk measure. Let $0 \le s < t \le T$. We consider the following properties:*

i) $\pi_t(X) \le \pi_t(Y) \Rightarrow \pi_s(X) \le \pi_s(Y)$

ii) $\pi_t(X) = \pi_t(Y) \Rightarrow \pi_s(X) = \pi_s(Y)$

iii) $\pi_s(\pi_t(X)) = \pi_s(X)$

iv) $\pi_t(X) \le 0 \Rightarrow \pi_s(X) \le 0$ *or* $\pi_t(X) \ge 0 \Rightarrow \pi_s(X) \ge 0$.

Property i) is the standard definition of time-consistency, while property iii) is commonly referred to as recursiveness. Properties i), ii) and iii) are known to be equivalent for dynamic monetary risk measures (see e.g., Föllmer and Schied (27)). As we show in the next lemma, they are also equivalent for monotone, conditionally positively homogeneous, normalized dynamic risk measures. We will therefore refer to i), ii) and iii) simply as time-consistency. Properties iv) are called weak acceptance consistency and weak rejection consistency, respectively.

Lemma 15. *Let $\pi_t \colon L^\infty_{++}(\mathcal{F}_T) \to L^\infty_{+}(\mathcal{F}_t)$ be a monotone and conditionally positively homogeneous dynamic risk measure with $\pi_t(1) = 1$. Then properties i), ii), and iii) are equivalent.*

Proof. Clearly, i) implies ii). From conditional positive homogeneity and the normalization $\pi_t(1) = 1$, it follows that $\pi_t(\pi_t(X)) = \pi_t(1 \cdot \pi_t(X)) = \pi_t(X)$, hence ii) implies iii). The implication iii) $\Rightarrow$ i) can be proved similarly to Lemma 11.11 of Föllmer and Schied (27), where the proof of that implication is based only on monotonicity. $\square$

It is straightforward to see that if $\Phi(x) = x^p$, $p \geq 1$, in Definition 2, then H_t^Φ is time-consistent. Indeed, in this case (see Example 4)

$$H_t^\Phi(X) = \left(\mathbb{E}[X^p|\mathcal{F}_t]\right)^{1/p},$$

so

$$H_s^\Phi(H_t^\Phi(X)) = \left(\mathbb{E}\left[\left(\mathbb{E}[X^p|\mathcal{F}_t]\right)|\mathcal{F}_s\right]\right)^{1/p} = H_s^\Phi(X).$$

A related property for a conditional risk measure is iterativity (see e.g., Bühlmann (8), Gerber (31), Haezendonck and Goovaerts (34)), which requires that for each sub σ-algebra $\mathcal{G} \subset \mathcal{F}$, it holds that

$$\pi(\pi_{\mathcal{G}}(X)) = \pi(X), \tag{10}$$

where in the present context $\pi_{\mathcal{G}}$ is defined as in (2). Different from time-consistency, iterativity does not refer to a particular filtration fixed in advance. Clearly, if $\Phi(x) = x^p$ with $p \geq 1$, then $H_{\mathcal{G}}^\Phi$ is iterative. In the next theorem we provide a direct proof that, in fact, the only conditional Orlicz premia that satisfy iterativity are conditional certainty equivalents with a power function Φ (i.e., conditional p-norms).

Theorem 16. *Let $H_{\mathcal{G}}^\Phi$ be as in (2). If $H_{\mathcal{G}}^\Phi$ is iterative in the sense of (10), then*

$$H_{\mathcal{G}}^\Phi(X) = \Phi^{-1}\left(\mathbb{E}[\Phi(X)|\mathcal{G}]\right), \tag{11}$$

with $\Phi(x) = x^p$ for some $p \geq 1$.

Proof. We first prove that if $H_{\mathcal{G}}^\Phi$ satisfies iterativity, then

$$H_0^\Phi(X) = \Phi^{-1}\left(\mathbb{E}[\Phi(X)]\right).$$

Let $X \in L_+^\infty(\mathcal{F})$. If $H_0^\Phi(X) = 1$, then by Proposition 3 item (i),

$$H_0^\Phi(X) = \Phi^{-1}(\mathbb{E}[\Phi(X)]).$$

If $H_0^\Phi(X) \neq 1$, then by the continuity of Φ there exists $z > 0$ and $q \in (0,1)$ such that

$$q\Phi(z) + (1-q)\Phi(H_0^\Phi(X)) = 1, \tag{12}$$

which implies that $H_0^\Phi(Z) = 1$, where Z is defined as follows:

$$Z := \begin{cases} z & \text{with prob. } q, \\ H_0^\Phi(X) & \text{with prob. } 1-q. \end{cases}$$

Since $(\Omega, \mathcal{F}, P)$ is nonatomic, there exists $A \in \mathcal{F}$ with $P(A) = q$ and $X' \in \mathcal{F}$ with $X' \overset{d}{=} X$ and X' independent of A. Letting $Y := z1_A + X'1_{A^c}$ and $\mathcal{G} := \{A, A^c, \Omega, \emptyset\}$, by items (g) and (l) in Proposition 3 and law invariance of H_0^Φ, it follows that

$$H_{\mathcal{G}}^\Phi(Y) = z1_A + H_{\mathcal{G}}^\Phi(X')1_{A^c} = z1_A + H_0^\Phi(X')1_{A^c} = z1_A + H_0^\Phi(X)1_{A^c},$$

so $H_{\mathcal{G}}^\Phi(Y) \overset{d}{=} Z$ and law invariance of H_0^Φ implies $H_0^\Phi(H_{\mathcal{G}}^\Phi(Y)) = H_0^\Phi(Z) = 1$. From iterativity, $H_0^\Phi(Y) = H_0^\Phi(H_{\mathcal{G}}^\Phi(Y)) = 1$, so

$$1 = \mathbb{E}[\Phi(Y)] = q\Phi(z) + (1-q)\,\mathbb{E}[\Phi(X')] = q\Phi(z) + (1-q)\,\mathbb{E}[\Phi(X)]. \tag{13}$$

Upon comparing (13) with (12) we get that

$$\mathbb{E}[\Phi(X)] = \Phi(H_0^\Phi(X)).$$

The result for a general t follows from conditional law invariance; the final part of the thesis follows as in the next theorem by the well known characterization of positively homogeneous certainty equivalents. □

A characterization result under the hypothesis of time-consistency is more difficult to obtain and requires additional hypotheses on the filtration $\mathcal{F}_t$. Following Kupper and Schachermayer (37), we assume that time is discrete (but note that a continuous-time dynamic risk measure can be embedded in this discrete-time setting) and that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\dots,T}, P)$ is a standard filtered probability space, i.e., it is isomorphic to $[0,1]^{\mathbb{N}}$ with its Borel σ-algebra and the product of Borel measures $\lambda^{\mathbb{N}}$. It turns out that in this setting the only dynamic Orlicz premia that satisfy time-consistency are also conditional certainty equivalents with a power function Φ, hence we obtain again an 'if and only if' characterization result.

Theorem 17. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,\dots,T}, P)$ be a standard filtered probability space and let $H_t^\Phi \colon L_+^\infty(\mathcal{F}_T) \to L_+^\infty(\mathcal{F}_t)$ be defined as in (1). If H_t^Φ is time-consistent, then*

$$H_t^\Phi(X) = \Phi^{-1}\left(\mathbb{E}_P\left[\Phi(X)\middle|\,\mathcal{F}_t\right]\right), \qquad \text{for any } X \in L_+^\infty(\mathcal{F}_T), \tag{14}$$

with $\Phi(x) = x^p$ for some $p \geq 1$.

Proof. To prove the result we will apply Theorem 1.4 of (37). We start by verifying that all its hypotheses are satisfied. First of all, H_0^Φ is real-valued. Furthermore, H_t^Φ satisfies constancy and locality (by Proposition 3 items (c) and (g)) and time-consistency by assumption. Law invariance of H_0^Φ is straightforward. The Fatou property of H_0^Φ has been established in Haezendonck and Goovaerts (34) and Goovaerts et al. (33) (see also Prop. 2(g) in Bellini and Rosazza Gianin (3)), while $\|\cdot\|_\infty$-continuity of H_0^Φ is a direct consequence of Proposition 3 items (b) and (f). Strict monotonicity of H_0^Φ follows from Lemma 3 in Bellini and Rosazza Gianin (3) since H_0^Φ on L_+^∞ (for $X \neq 0$) can be seen as the unique solution to the equation $\mathbb{E}_P\left[\Phi\left(X/H_0^\Phi\right)\right] = 1$. All the hypotheses of Theorem 1.4 of (37) are then satisfied on H_0^Φ restricted to $L_{++}^\infty(\mathcal{F}_T)$.

By applying Theorem 1.4 of Kupper and Schachermayer (37), it follows that H_t^Φ is of the form in (14) for some strictly increasing and continuous ℓ and $L_{++}^\infty(\mathcal{F}_T)$. Moreover, ℓ reduces to a power function or a logarithmic function because of positive homogeneity of H_t^Φ and by the De Finetti-Nagumo-Kolmogorov Theorem on the characterization of expected utility functional (see De Finetti (17), and also Frittelli (28) and Laeven and Stadje (38)). Subadditivity of H_t^Φ excludes the logarithmic case (see, for instance, Example 2 in Bellini et al. (5)).

The case of H_0^Φ on the whole $L_+^\infty(\mathcal{F}_T)$ can be obtained by continuity. Assume indeed that X takes the value 0 in $A \in \mathcal{F}_T$. Then, $X_n = \frac{1}{n}1_A + X1_{A^c} \to X$ in L^∞, so by $\|\cdot\|_\infty$-continuity it follows that $H_0^\Phi(X_n) \to H_0^\Phi(X)$. Once ℓ is extended with continuity at 0, for $t = 0$ (14) is true for any $X \in L_+^\infty(\mathcal{F}_T)$.

By considering the value of H_0^Φ on Bernoulli random variables, it is easy to check that indeed $\ell = \Phi$. The case of a general t follows from time-consistency. □

Let us now consider the case of dynamic robust Orlicz premia, defined by (7). If $\Phi(x) = x^p$ for $p \geq 1$ and if ρ_t is positively homogeneous as in (6), then time-consistency of $H_t^{\Phi,\rho}$ is equivalent to time-consistency of ρ_t.

Lemma 18. *Let $H_t^{\Phi,\rho}$ be as in (7) with $\Phi(x) = x^p$ for $p \geq 1$ and let ρ_t be positively homogeneous. Then $H_t^{\Phi,\rho}$ is time-consistent if and only if ρ_t is time-consistent.*

Proof. Under these hypotheses, it holds that

$$H_t^{\Phi,\rho}(X) = (\rho_t(X^p))^{1/p} \text{ and } \rho_t(X) = (H_t^{\Phi,\rho}(X^{1/p}))^p.$$

If ρ_t is time-consistent, then

$$\begin{aligned} H_s^{\Phi,\rho}(H_t^{\Phi,\rho}(X)) = H_s^{\Phi,\rho}((\rho_t(X^p))^{1/p}) = (\rho_s\left(\rho_t\left(X^p\right)\right))^{1/p} = \\ = (\rho_s\left(X^p\right))^{1/p} = H_s^{\Phi,\rho}(X), \end{aligned}$$

which shows time-consistency of $H_t^{\Phi,\rho}$. Similarly, if $H_t^{\Phi,\rho}$ is time-consistent, then

$$\begin{aligned} \rho_s(\rho_t(X)) = \rho_s((H_t^{\Phi,\rho}(X^{1/p}))^p) = (H_s^{\Phi,\rho}(H_t^{\Phi,\rho}(X^{1/p})))^p = \\ = (H_s^{\Phi,\rho}(X^{1/p}))^p = \rho_s(X). \end{aligned}$$

□

Time-consistency of ρ_t of the form (5) is well studied in the literature; see e.g., Bion-Nadal (6; 7) for a characterization via an additive cocycle property of the penalty function. Time-consistency of ρ_t of the form (6) is less explored.

The following results provide some connections (or 'inheritance relations') between time-consistency properties of dynamic robust Orlicz premia $H_t^{\Phi,\rho}$ and the corresponding dynamic robust Haezendonck-Goovaerts risk measures $HG_t^{\Phi,\rho}$. In order to prove time-consistency of $HG_t^{\Phi,\rho}$, by Bion-Nadal (6; 7) and Delbaen (19), it suffices to verify m-stability of the set of generalized scenarios in its dual representation.

Proposition 19. *Let $H_t^{\Phi,\rho}$ be a dynamic robust Orlicz premium as in (7) and let ρ_t satisfy the Lebesgue property. If, for each $s, t \in [0,T]$ with $s \leq t$ and for any $X \in L^\infty(\mathcal{F}_T)$ it holds that*

$$H_s^{\Phi,\rho}(H_t^{\Phi,\rho}(X)) \leq H_s^{\Phi,\rho}(X), \tag{15}$$

then the corresponding dynamic robust risk measure $HG_t^{\Phi,\rho}$ in (8) is time-consistent.

Proof. Denote by $HG^{\Phi,\rho}_{t,u}$ the restriction of $HG^{\Phi,\rho}_t$ to $L^\infty(\mathcal{F}_u)$. By Proposition 13, $HG^{\Phi,\rho}_{t,u}$ has the following dual representation:

$$HG^{\Phi,\rho}_{t,u}(X) = \operatorname*{ess\,sup}_{R \in \mathcal{R}_{t,u}} \mathbb{E}_R \left[X | \, \mathcal{F}_t \right], \quad X \in L^\infty(\mathcal{F}_u),$$

where

$$\mathcal{R}_{t,u} = \{ R \in \mathcal{P}_{t,u} \mid \mathbb{E}_R \left[Z | \, \mathcal{F}_t \right] \leq H^{\Phi,\rho}_{t,u}(Z^+) \text{ for any } Z \in L^\infty(\mathcal{F}_u) \} \tag{16}$$

and $\mathcal{P}_{t,u}$ is the subset of probability measures in $\mathcal{P}_t$ that are defined on $(\Omega, \mathcal{F}_u)$. By Bion-Nadal (6; 7) and Delbaen (19), in order to prove time-consistency of $HG^{\Phi,\rho}_t$ it is sufficient to verify the m-stability of the dual sets $(\mathcal{R}_{t,u})_{0 \leq t \leq u \leq T}$. Let $0 \leq s \leq t \leq u \leq T$ and let $R_1 \in \mathcal{R}_{s,t}$, $R_2 \in \mathcal{R}_{t,u}$. Denote by $\bar{R}$ the pasting between R_1 and R_2. To prove that $\bar{R}$ belongs to $\mathcal{R}_{s,u}$, notice that from (15) it follows that for any $Y \in L^\infty(\mathcal{F}_u)$

$$\begin{aligned} \mathbb{E}_{\bar{R}} \left[Y \mid \mathcal{F}_s \right] &= \mathbb{E}_{R_1} \left[\mathbb{E}_{R_2} \left[Y \mid \mathcal{F}_t \right] | \, \mathcal{F}_s \right] \leq \mathbb{E}_{R_1} [H^{\Phi,\rho}_t(Y^+) \mid \mathcal{F}_s] \\ &\leq H^{\Phi,\rho}_s(H^{\Phi,\rho}_t(Y^+)) \leq H^{\Phi,\rho}_s(Y^+), \end{aligned}$$

which implies that $\bar{R} \in \mathcal{R}_{s,u}$. □

In particular, the hypothesis (15) in Proposition 19 is satisfied if $H^{\Phi,\rho}_t$ is time-consistent. A weaker version of the thesis holds in much more general situations, as the following proposition shows.

Proposition 20. *Let $(H_t)_{t \in [0,T]}$ be any family of functionals $H_t \colon L^\infty_+(\mathcal{F}_T) \to L^\infty_+(\mathcal{F}_t)$ satisfying property (15), constancy and cash-subadditivity. Then the corresponding dynamic risk measure $(HG_t)_{t \in [0,T]}$ defined as in (8) is weakly rejection consistent in the sense of Definition 14, that is, for any $0 \leq s \leq t \leq T$,*

$$HG_t(X) \geq 0 \Rightarrow HG_s(X) \geq 0.$$

Proof. Suppose that $HG_t(X) \geq 0$ for some $X \in L^\infty(\mathcal{F}_T)$. By (8) it follows that

$$x_t + H_t \left((X - x_t)^+ \right) \geq 0, \quad \forall x_t \in L^\infty(\mathcal{F}_t),$$

hence, for any $s \leq t$,

$$x_s + H_t \left((X - x_s)^+ \right) \geq 0, \quad \forall x_s \in L^\infty(\mathcal{F}_s). \tag{17}$$

Fix now any $s \leq t$. We are going to prove that $x_s + H_s\left((X - x_s)^+\right) \geq 0$ for any $x_s \in L^\infty(\mathcal{F}_s)$, hence $HG_s(X) \geq 0$. By assumption (15) and constancy of H_t, for any $x_s \in L^\infty_-(\mathcal{F}_s)$ inequality (17) becomes

$$\begin{aligned}
&H_s(H_t((X - x_s)^+)) \geq H_s(-x_s) \\
&H_s((X - x_s)^+) \geq -x_s \\
&x_s + H_s((X - x_s)^+) \geq 0. \qquad (18)
\end{aligned}$$

A fortiori, inequality (18) is satisfied also for any $x_s \in L^\infty_+(\mathcal{F}_s)$. For an arbitrary $x_s \in L^\infty(\mathcal{F}_s)$, set $A = \{x_s \geq 0\} \in \mathcal{F}_s$. Inequality (17) implies therefore that

$$x_s 1_A + H_t((X - x_s)^+) \geq -x_s 1_{A^c}.$$

Proceeding as above, we get

$$\begin{aligned}
&H_s(x_s 1_A + H_t((X - x_s)^+)) \geq H_s(-x_s 1_{A^c}) \\
&H_s(x_s 1_A) + H_s(H_t((X - x_s)^+)) \geq -x_s 1_{A^c} \qquad (19) \\
&x_s 1_A + H_s((X - x_s)^+) \geq -x_s 1_{A^c} \qquad (20) \\
&x_s + H_s((X - x_s)^+) \geq 0,
\end{aligned}$$

where (19) follows from subadditivity and constancy, while (20) follows from assumption (15) and constancy of H_s. From the arguments above it follows that $x_s + H_s((X - x_s)^+) \geq 0$ holds for any $x_s \in L^\infty(\mathcal{F}_s)$, hence $HG_s(X) \geq 0$. □

The following result is a partial converse of Proposition 20. We omit the proof, which follows immediately by (8).

Proposition 21. *Let $(H_t)_{t \in [0,T]}$ be a family of functionals $H_t : L^\infty_+(\mathcal{F}_T) \to L^\infty_+(\mathcal{F}_t)$ satisfying monotonicity and constancy. If the corresponding dynamic risk measure HG_t satisfies weak rejection consistency in the sense of Definition 14 item v), then if for each $x_t \in L^\infty(\mathcal{F}_t)$ it holds that*

$$H_t\left((Y - x_t)^+\right) \geq H_t\left((-x_t)^+\right),$$

then for each $x_s \in L^\infty(\mathcal{F}_s)$, $0 \leq s \leq t$, it holds that

$$H_s\left((Y - x_s)^+\right) \geq H_s\left((-x_s)^+\right).$$

Notice that the thesis of Proposition 21 is weaker than and is implied by time-consistency of H_t. Indeed,

$$H_t\left((Y-x_t)^+\right) \geq H_t\left((-x_t)^+\right) \text{ for any } x_t \in L^\infty(\mathcal{F}_t)$$

implies that

$$H_t\left((Y-x_s)^+\right) \geq H_t\left((-x_s)^+\right) \text{ for any } x_s \in L^\infty(\mathcal{F}_s),$$

and by monotonicity and time-consistency of H_s it follows that

$$H_s\left((Y-x_s)^+\right) = H_s\left(H_t\left(Y-x_s)^+\right)\right) \geq H_s\left(H_t\left((-x_s)^+\right)\right) = H_s\left((-x_s)^+\right),$$

for any $x_s \in L^\infty(\mathcal{F}_s)$.

7. Conclusions

In this paper we have focused on introducing and developing the mathematical and probabilistic theory of dynamic return risk measures, and dynamic Orlicz premia and Haezendonck-Goovaerts (HG) risk measures in particular. An interesting and relevant related problem is that of statistical inference for dynamic return risk measures. The results presented in this paper naturally precede, and can serve as a starting point for the development of, such statistical inference techniques. This resembles the static case where the theory of return risk measures, and Orlicz premia and HG-risk measures in particular, has been developed since the eighties of the previous century, and much more recently their statistical inference has been studied in several papers (e.g., Ahn and Shyamalkumar (1), Liu, Peng and Wang (40), Mao and Hu (42), Peng, Wang and Zheng (43), Wang and Peng (51)). We believe the development of statistical inference techniques for dynamic return risk measures is a promising future research avenue.

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